RATIONAL TORUS-EQUIVARIANT STABLE HOMOTOPY I: CALCULATING GROUPS OF STABLE MAPS.

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Abstract. We construct an abelian category $\mathcal{A}(G)$ of sheaves over a category of closed subgroups of the r-torus G. The category $\mathcal{A}(G)$ is of injective dimension r, and can be used as a model for rational G-spectra. Indeed, we show that there is a homology theory

$$\pi_*^{\mathcal{A}} : G\text{-}\mathbf{spectra} \longrightarrow \mathcal{A}(G)$$

on rational G-spectra with values in $\mathcal{A}(G)$ and the associated Adams spectral sequence converges for all rational G-spectra and collapses at a finite stage.

This is the first paper in a series of three. It culminates in [11] where the author and B.E.Shipley combine the Adams spectral sequence constructed here with the enriched Morita equivalence of Schwede and Shipley [13] to deduce that the category of differential graded objects of $\mathcal{A}(G)$ is Quillen equivalent to the category of rational G-spectra.

Contents

I am grateful to J.P.May and B.E.Shipley for numerous comments on early drafts of this paper.

Part 1. **Introduction.**

1. Summary.

1.A. **The context.** Non-equivariantly, rational stable homotopy theory is very simple: the homotopy category of rational spectra is equivalent to the category of graded rational vector spaces, and all cohomology theories are ordinary. The author has conjectured that for each compact Lie group G, there is an abelian category $\mathcal{A}(G)$ so that the homotopy category of rational G-spectra is equivalent to the derived category of $\mathcal{A}(G)$:

$$\mathrm{Ho}(G\text{-spectra}/\mathbb{Q}) \simeq D(\mathcal{A}(G)).$$

The conjecture describes various properties of $\mathcal{A}(G)$, and in particular asserts that its injective dimension is equal to the rank of G. Thus one can expect to make complete calculations in rational equivariant stable homotopy theory, and one can classify cohomology theories. Indeed, one can construct a cohomology theory by writing down an object in $\mathcal{A}(G)$: this is how $SO(2)$-equivariant elliptic cohomology was constructed in [8], and work on curves of higher genus is underway.

The conjecture is elementary for finite groups where $\mathcal{A}(G) = \prod_{(H)} \mathbb{Q}W_G(H)$-mod [10], so that all cohomology theories are again ordinary. The conjecture has been proved for the rank 1 groups $G = SO(2), O(2), SO(3)$ in [6, 5, 7], where $\mathcal{A}(G)$ is more complicated. It is natural to go on to conjecture that the equivalence comes from a Quillen equivalence

$$G\text{-spectra}/\mathbb{Q} \simeq dg\mathcal{A}(G),$$

for suitable model structres, and Shipley has established this for $G = SO(2)$ [14]. The present paper is the first in a series [9, 11] establishing this Quillen equivalence, and hence the derived equivalence, when G is the r-dimensional torus for any $r \geq 0$.

1.B. **The results.** The purpose of the present paper is to provide a means for calculation in the homotopy category of rational G-spectra, where, for the rest of the paper, G is an r-dimensional torus. In Part 2 we construct an abelian category $\mathcal{A}(G)$ (Definition 3.9) and in 5.3 show it is of finite injective dimension. The category $\mathcal{A}(G)$ is a category of sheaves M on the space of subgroups of G, and the value $M(U)$ of a sheaf on a set U of subgroups captures the information about spaces with isotropy groups in U. In Part 3 we construct a homology theory

$$\pi_*^{\mathcal{A}} : G\textbf{-spectra} \longrightarrow \mathcal{A}(G)$$

with values in $\mathcal{A}(G)$, and show that it is an effective calculational tool in that there is an Adams spectral sequence. The main theorem is as follows.

Theorem 1.1. *There is a spectral sequence*

$$\mathrm{Ext}^{*,*}_{\mathcal{A}(G)}(\pi_*^{\mathcal{A}}(X), \pi_*^{\mathcal{A}}(Y)) \Rightarrow [X, Y]^G_*,$$

strongly convergent for all G-spectra X and Y.

The category $\mathcal{A}(G)$ is of injective dimension r, and so the spectral sequence is concentrated between rows 0 and r: it therefore collapses at the E_{r+1}-page.

The special case $r = 1$ provided the basis for the results of [6]. In addition to being a powerful tool, it is a perfectly practical one, since it is straightforward to make calculations in $\mathcal{A}(G)$. The theorem essentially describes the category of rational G-spectra up to a finite filtration. For many purposes this is quite sufficient, but the other papers in the series go further. In [11] Shipley and the author combine the Adams spectral spectral sequence of the present paper with the work of Schwede and Shipley [13] to show that the category of rational G-spectra is Quillen equivalent to $dg\mathcal{A}(G)$. The paper [9] provides the information about the algebraic structure of the category $\mathcal{A}(G)$ required in [11], and establishes that the injective dimension of $\mathcal{A}(G)$ is precisely r. The principal results of the present paper were proved by 2001, but publication was delayed until the form of results required by [11] was clear.

Convention 1.2. Certain conventions are in force throughout the paper and the series. The most important is that *everything is rational*: all spectra and homology theories are rationalized without comment. The second is the standard one that 'subgroup' means 'closed subgroup'. We attempt to let inclusion of subgroups follow the alphabet, so that when there are inclusions they are in the pattern $G \supseteq H \supseteq K \supseteq L$. The other convention beyond the usual one that H_1 denotes the identity component of H is that $\tilde{H}$ denotes a subgroup with identity component H and $\hat{H}$ denotes a subgroup in which H is cotoral (i.e., so that $H \subseteq \hat{H}$ and $\hat{H}/H$ is a torus).

Finally, cohomology is unreduced unless indicated to the contrary with a tilde, so that $H^*(BG/K) = \tilde{H}^*(BG/K_+)$ is the unreduced cohomology ring.

1.C. **Some standard constructions.** We recall some standard constructions from equivariant homotopy theory. As a general reference on stable equivariant homotopy theory, we use [12].

For any family $\mathcal{H}$ of subgroups (i.e., a collection closed under passage to conjugates and smaller subgroups), we may consider the G-space $E\mathcal{H}$, which is universal amongst $\mathcal{H}$-spaces; it is characterized by the fact that its K-fixed points are either empty (if $K \not\in \mathcal{H}$) or contractible (if $K \in \mathcal{H}$). We then have a basic cofibre sequence

$$E\mathcal{H}_+ \longrightarrow S^0 \longrightarrow \tilde{E}\mathcal{H},$$

and $\tilde{E}\mathcal{H}$ may also be constructed as the join $S^0 * E\mathcal{H}$.

The families playing a significant role for us are the family $\mathcal{F}$ of all finite subgroups of G and the family $[\not\supseteq K]$ of subgroups not containing K. The role of $[\not\supseteq K]$ is worth further comment. If X is a G-space and we write $\Phi^K X$ for its K-fixed points, it is clear that the inclusion $\Phi^K X \longrightarrow X$ induces an equivalence

$$\tilde{E}[\not\supseteq K] \wedge X \simeq \tilde{E}[\not\supseteq K] \wedge \Phi^K X.$$

In fact Φ^K admits an extension to a functor $\Phi^K : G$-**spectra** $\longrightarrow G/K$-**spectra** (known as the *geometric* fixed point functor), and the same equivalence holds.

This holds for any group G, provided K is normal, but for an abelian group there is a particularly convenient construction of $\tilde{E}[\not\supseteq K]$. Indeed, if we define

$$S^{\infty V(K)} = \lim_{\rightarrow\, V^K=0} S^V,$$

the special from of G means

$$\tilde{E}[\not\supseteq K] \simeq S^{\infty V(K)}.$$

(If L does not contain K then we can find a representation V with $V^K = 0$ and $V^L \neq 0$. Indeed, K has non-trivial image in G/L, and hence G/L has a character α with $\alpha^K = 0$). This construction using spheres gives a connection with Euler classes and thence to commutative algebra. This is a key ingredient in the results.

Choosing an orientation of ordinary cohomology, we have Euler classes of vector bundles. Thus if W is a representation of G/K then there is an associated Euler class $c_H(W) \in H^{|W|}(BG/K)$. These enter the picture since $c_H(W)$ is the pullback of the Thom class $\tau_H(W)$ along the inclusion $S^0 \longrightarrow S^W$:

$$\begin{array}{ccc} \tilde{H}^{|W|}(EG/K_+ \wedge_{G/K} S^W) & \longrightarrow & \tilde{H}^{|W|}(EG/K_+ \wedge_{G/K} S^0) \\ \tau_H(W) & \longmapsto & c_H(W). \end{array}$$

1.D. **Outline of the argument.** First we must construct the the category $\mathcal{A}(G)$. This is a category of sheaves on the space of subgroups of G. In fact we consider the 'natural' open sets

$$U(K) = \{H \mid H \supseteq K\}$$

of isotropy groups, where K runs through the *connected* subgroups of G, and an object M of $\mathcal{A}(G)$ is specified by its values $M(U(K))$ and the restriction maps $M(U(K)) \longrightarrow M(U(H))$ when $U(K) \supseteq U(H)$ (i.e., when $K \subseteq H$). These are required to satisfy certain conditions that we explain shortly, but $M(U(K))$ contains information about isotropy groups H in $U(K)$.

It is rather easy to write down the functor $\pi_*^{\mathcal{A}}(X)$.

Definition 1.3. For a G-spectrum X we define $\pi_*^{\mathcal{A}}(X)$ on U-open subsets by

$$\pi_*^{\mathcal{A}}(X)(U(K)) = \pi_*^G(DE\mathcal{F}_+ \wedge S^{\infty V(K)} \wedge X).$$

Here $E\mathcal{F}_+$ is the universal space for the family $\mathcal{F}$ of finite subgroups with a disjoint basepoint added and $DE\mathcal{F}_+ = F(E\mathcal{F}_+, S^0)$ is its functional dual (the function spectrum of maps from $E\mathcal{F}_+$ to S^0). Note that since G-space $S^{\infty V(K)}$ is defined by

$$S^{\infty V(K)} = \lim_{\rightarrow\ V^K=0} S^V,$$

when $K \subseteq H$ there is a map $S^{\infty V(K)} \longrightarrow S^{\infty V(H)}$ inducing the restriction map $\pi_*^{\mathcal{A}}(X)(U(K)) \longrightarrow \pi_*^{\mathcal{A}}(X)(U(H))$. □

The objects of $\mathcal{A}(G)$ have the structure of modules over the structure sheaf $\mathcal{O}$ introduced formally in Subsection 3.C. The definition of the structure sheaf is based on the ring

$$\mathcal{O}_{\mathcal{F}} = \prod_{F \in \mathcal{F}} H^*(BG/F_+),$$

where the product is over the family $\mathcal{F}$ of finite subgroups of G. For this we use Euler classes: indeed if V is a representation of G we may defined $c(V) \in \mathcal{O}_{\mathcal{F}}$ by taking its components $c(V)(F) = c_H(V^F) \in H^*(BG/F_+)$ to be classical ordinary homology Euler classes.

The sheaf $\mathcal{O}$ is defined by

$$\mathcal{O}(U(K)) = \mathcal{E}_K^{-1}\mathcal{O}_{\mathcal{F}}$$

where $\mathcal{E}_K = \{c(V) \mid V^K = 0\} \subseteq \mathcal{O}_{\mathcal{F}}$ is the multiplicative set of Euler classes of K-essential representations.

To see that $\pi_*^{\mathcal{A}}(X)$ is a module over $\mathcal{O}$, the key is to understand S^0.

Theorem 1.4. *The image of S^0 in $\mathcal{A}(G)$ is the structure sheaf:*

$$\mathcal{O} = \pi_*^{\mathcal{A}}(S^0).$$

We prove this in the course of Sections 6 to 9.

There are then two requirements on $\mathcal{O}$-modules to be objects of $\mathcal{A}(G)$. Firstly they must be *quasi-coherent*, in that

$$M(U(K)) = \mathcal{E}_K^{-1} M(U(1)),$$

where $\mathcal{E}_K$ is the set of Euler classes of K-essential representations as before. The definition of $\pi_*^{\mathcal{A}}(X)$ shows that quasi-coherence for $\pi_*^{\mathcal{A}}(X)$ is just a matter of understanding Euler classes, which we do in Section 8. The second condition involves the ring $\mathcal{O}_{\mathcal{F}}$ and its analogue

$$\mathcal{O}_{\mathcal{F}/K} = \prod_{\tilde{K} \in \mathcal{F}/K} H^*(BG/\tilde{K}_+)$$

for the quotient modulo a connected subgroup K, where $\mathcal{F}/K$ is the family of subgroups $\tilde{K}$ of G with identity component K. The second condition is that the object should be *extended*, in the sense that there is a specified isomorphism

$$M(U(K)) = \mathcal{E}_K^{-1} \mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}/K}} \phi^K M$$

for some $\mathcal{O}_{\mathcal{F}/K}$-module $\phi^K M$. The extendedness of $\pi_*^{\mathcal{A}}(X)$ follows from a construction of the geometric fixed point functor, and it turns out that

$$\phi^K \pi_*^{\mathcal{A}(G)}(X) = \pi_*^{G/K}(DE\mathcal{F}_+ \wedge \Phi^K(X)),$$

where Φ^K is the geometric fixed point functor. This sketches the proof of the following result, proved formally in Section 9.

Corollary 1.5. *The functor $\pi_*^{\mathcal{A}}$ takes values in the abelian category $\mathcal{A}(G)$.*

This outlines the construction of the functor $\pi_*^{\mathcal{A}}$. To construct the Adams spectral sequence we need to realize an injective resolution of $\pi_*^{\mathcal{A}}(X)$ in $\mathcal{A}(G)$, and to prove the Adams spectral sequence works for maps into an injective. We must therefore first realize sufficiently many injectives. We show that there is a right adjoint f_K to the evaluation at K functor ϕ^K (described in detail in Subsection 4.A). Thus for a suitable module N over $\mathcal{O}_{\mathcal{F}/K}$ we may form an object $f_K(N)$ in $\mathcal{A}(G)$. Taking $N = H_*(BG/\tilde{K})$ for a subgroup $\tilde{K}$ with identity component K, viewed as an $\mathcal{O}_{\mathcal{F}/K}$-module via projection onto $H^*(BG/\tilde{K})$, we obtain the object $I(\tilde{K}) = f_K(H_*(BG/\tilde{K}))$. This is injective since $H_*(BG/\tilde{K})$ is injective over $H^*(BG/\tilde{K})$. It turns out (Lemma 10.2) that a suspension of $I(\tilde{K})$ is realized by the G-spectrum $E\langle\tilde{K}\rangle$ defined in 7.1 in the sense that

$$I(\tilde{K}) = \pi_*^{\mathcal{A}}(\Sigma^{-c} E\langle\tilde{K}\rangle)$$

where $\tilde{K}$ is of codimension c. Next we need to understand maps into injectives, showing that

$$\pi_*^{\mathcal{A}} : [X, I]_*^G \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{A}}(\pi_*^{\mathcal{A}}(X), \pi_*^{\mathcal{A}}(I))$$

for these sufficiently many injectives I. This constructs a spectral sequence with the correct E_2-term. Finally we must show convergence by showing that $\pi_*^{\mathcal{A}}(X) = 0$ implies $X \simeq *$. This is an easy consequence of the geometric fixed point Whitehead theorem 10.4.

2. Formal behaviour of equivariant homology theories.

There are two ways one may hope to encode data about the homology of fixed point sets. They are close enough to be confusing, so it is worth making them explicit at the outset. We consider the case that G is a torus, and the reader may want to bear in mind the examples of stable homotopy and K-theory.

Given a G-equivariant homology theory $\tilde{F}^G_*(\cdot)$ and a G-space X we may consider the system of values

$$H \longmapsto \tilde{F}^{G/H}_*(\Phi^H X),$$

where $\Phi^H X$ denotes the (geometric) H-fixed point set of X. If $K \subseteq H$ there is an inclusion $\Phi^H X \longrightarrow \Phi^K X$ of G/K-spaces and hence a map $\tilde{F}^{G/K}_*(\Phi^H X) \longrightarrow \tilde{F}^{G/K}_*(\Phi^K X)$, but in general there will not be a map $\tilde{F}^{G/H}_*(\Phi^H X) \longrightarrow \tilde{F}^{G/K}_*(\Phi^K X)$. However in favourable circumstances there are maps of this sort, and we accordingly call a *contravariant* functor M on subgroups an *inflation functor*. If we are just given the values $\tilde{F}^{G/H}_*(\Phi^H X)$ and no structure maps between them we refer to an *inflation system*. Because of the variance and the motivation we sometimes write $M(G/H)$ to suggest dependence on the quotient group G/H.

On the other hand, we may always consider

$$X \wedge \tilde{E}[\not\supseteq H] \simeq \Phi^H X \wedge \tilde{E}[\not\supseteq H]$$

where $[\not\supseteq H]$ is the family of subgroups not containing H. If $K \subseteq H$ then there is a natural map $\tilde{E}[\not\supseteq K] \longrightarrow \tilde{E}[\not\supseteq H]$, and hence a map

$$\tilde{F}^G_*(\Phi^K X \wedge \tilde{E}[\not\supseteq K]) \longrightarrow \tilde{F}^G_*(\Phi^H X \wedge \tilde{E}[\not\supseteq H]).$$

Now,

$$\tilde{E}[\not\supseteq K] = \lim_{\substack{\rightarrow \\ V^K=0}} S^V,$$

and, under orientability hypotheses, $\tilde{F}^G_*(\Phi^K X \wedge \tilde{E}[\not\supseteq K])$ may be expressed as a localization $\mathcal{E}_K^{-1}\tilde{F}^G_*(X)$ of $\tilde{F}^G_*(X)$, where $\mathcal{E}_K$ is some multiplicatively closed subset of F^G_*, generated by "Euler classes" $e(V)$ with $V^K = 0$. We will call a *covariant* functor on subgroups of this form a *localization functor*.

Two major differences should be emphasized. First, inflation functors are contravariant in the subgroup whilst localization functors are covariant. Second, a localization functor takes values which are modules over F^G_*, whereas an inflation functor typically does not.

When we are fortunate enough that F gives an inflation functor and also has a localization theorem it may happen that the two structures are related in the sense that

$$\tilde{F}^G_*(\Phi^K X \wedge \tilde{E}[\not\supseteq K]) = \mathcal{E}_K^{-1} F^G_* \otimes_{F^{G/K}_*} \tilde{F}^{G/K}_*(\Phi^K X).$$

In other words, the favourable case is when we have the following structure, which will be properly defined and axiomatized in later sections.

1. R, a ring-valued inflation functor (such as $K \longmapsto F^{G/K}_*$)
2. M an inflation system, which is module valued functor over R, (such as $K \longmapsto \tilde{F}^{G/K}_*(\Phi^K X)$)
3. LM a localization functor, which is module valued over $R(G/1)$, (such as $K \longmapsto \tilde{F}^G_*(\Phi^K X \wedge \tilde{E}[\not\supseteq K])$, and

(4) an isomorphism

$$LM(K) = \mathcal{E}_K^{-1} R(G/1) \otimes_{R(G/K)} M(G/K).$$

In this case we say that the localization functor LM is *extended* with associated inflation system M. However, be warned that, even if M is an inflation functor (i.e., it has contravariant structure maps), this does not supply the structure maps for $\mathcal{E}_K^{-1} R(G/1) \otimes_{R(G/K)} M(G/K)$, so that LM requires further data.

Part 2. Categories of U-sheaves.

The objects of the abelian category $\mathcal{A}(G)$ are sheaves of modules over a sheaf $\mathcal{O}$ of rings. Accordingly we begin Section 3 by describing the inflation functor on which the structure sheaf $\mathcal{O}$ is based; we can then introduce Euler classes and proceed with the definition. Once $\mathcal{A}(G)$ is defined, we begin to control it in Section 4: first we import objects from module categories, and then show that these suffice to build all the objects and prove that $\mathcal{A}(G)$ has finite injective dimension.

3. The standard abelian category.

The present section leads up to the definition of the standard model as a certain category of U-sheaves of $\mathcal{O}$-modules. Before we can express the definition we need to introduce the structure sheaf $\mathcal{O}$, and before we can do this (Subsection 3.C) we need to describe its associated inflation functor (Subsection 3.A) and Euler classes (Subsection 3.B).

3.A. **The fundamental inflation functor.** The entire structure we discuss is founded on the inflation functor described in this section. We let ConnSub(G) denote the category of connected subgroups of G and inclusions. An inflation functor is a contravariant functor

$$M : \mathrm{ConnSub}(G) \longrightarrow \mathbf{AbGp}$$

We write $M_{G/H}$ for its value on H. The purpose of this section is to introduce a ring valued inflation functor

$$\mathcal{O}_{\mathcal{F}} : \mathrm{ConnSub}(G) \longrightarrow \mathbf{Rings},$$

whose value at K is written $\mathcal{O}_{\mathcal{F}/K}$. Other notations can be convenient and have been used elsewhere, for example $\mathcal{O}_{\mathcal{F}/K} = \mathcal{O}_{\mathcal{F}(K)} = \mathcal{O}(K) = R_{G/K}$, but we will stick to the above notation in this series.

For any connected subgroup K, we let

$$\mathcal{F}/K = \{\tilde{K} \mid K \text{ of finite index in } \tilde{K}\}$$

denote the set of subgroups of G with identity component K, which is in natural correspondence with the finite subgroups of G/K. Now take

$$\mathcal{O}_{\mathcal{F}/K} = \prod_{\tilde{K} \in \mathcal{F}/K} H^*(BG/\tilde{K})$$

where the product is over the set of subgroups with identity component K.

To describe the inflation maps, suppose K and L are connected and $L \subseteq K$ and L is of finite index in $\tilde{L}$. The inclusion defines a quotient map $q : G/L \longrightarrow G/K$ and hence

$$q_* : \mathcal{F}/L \longrightarrow \mathcal{F}/K.$$

The inflation map $\mathcal{O}_{\mathcal{F}/K} \longrightarrow \mathcal{O}_{\mathcal{F}/L}$ has $\tilde{L}$th component

$$\mathcal{O}_{\mathcal{F}/K} = \prod_{\tilde{K}\in\mathcal{F}/K} H^*(BG/\tilde{K}) \longrightarrow H^*(BG/q_*\tilde{L}) \longrightarrow H^*(BG/\tilde{L})$$

given by projection onto the term $H^*(BG/q_*\tilde{L})$ followed by the inflation map induced by the quotient $G/\tilde{L} \longrightarrow G/q_*\tilde{L}$.

Now an *inflation system* of $\mathcal{O}_{\mathcal{F}}$-modules is given by specifying an $\mathcal{O}_{\mathcal{F}/K}$-module $M_{G/K}$ for each subgroup K. No structure maps relating these modules are required.

3.B. **Euler classes.** We are now in a position to describe the Euler classes which are used in the localization process. This will allow us to discuss localization functors, and hence quasi-coherent and extended U-sheaves.

The Euler class of an arbitrary representation is defined in terms of those of simple representations using the product formula $e(V \oplus W) = e(V)e(W)$. Since G is abelian, it is enough to define Euler classes $e(\alpha) \in \mathcal{O}_{\mathcal{F}}$ for one dimensional representations α. We take

$$e(\alpha) \in \mathcal{O}_{\mathcal{F}} = \prod_{F\in\mathcal{F}} H^*(BG/F),$$

to be defined by

$$e(\alpha)(F) = \begin{cases} 1 & \text{if } \alpha^F = 0 \\ c_1(\alpha) & \text{if } \alpha \text{ is trivial on } F. \end{cases}$$

This is not a homogeneous element. The best way to sanitize this is to introduce an invertible sheaf associated to a representation (corresponding to suspension), and make $e(\alpha)$ a section of that. Thus $e(\alpha)$ should be thought of as a section of a line bundle vanishing at a finite group F if and only if F acts trivially on α. Since H acts trivially on α if and only if all finite subgroups of H act trivially, we can think of $e(\alpha)$ as defining the 'U-closed' set of subgroups of $\ker(\alpha)$.

There are enough representations of G in the sense that if H is fixed, it is separated from all subgroups (except those containing it) by an Euler class: if $H \not\subseteq K$ there is a representation α trivial over K and non-trivial over H. Accordingly, if H is connected, the open set $U(H)$ of subgroups containing H is defined by inverting the set

$$\mathcal{E}_H = \{e(W) \mid W^H = 0\}$$

of Euler classes of representations generated by characters not arising from G/H. If $\tilde{H}$ has identity component H we let $\mathcal{E}_{\tilde{H}} = \mathcal{E}_H$.

Example 3.1. For example if G is the circle group and z is the natural representation, $e(z)$ is supposed to define $\{1\}$. We think of $e(z)$ as the function (or rather global section) given on finite subgroups by

$$e(z)(F) = \begin{cases} c & \text{if } F = 1 \\ 1 & \text{if } F \neq 1. \end{cases}$$

Remark 3.2. The correspondence with divisors can be very important (see [8]). By definition $e(\alpha)$ vanishes to the first order at finite subgroups of $\ker(\alpha)$. It is thus natural to view the line bundle of which $e(\alpha)$ is a generating section as corresponding to the 'divisor' $\overline{\ker(\alpha)}$, and call it $\mathcal{O}(\ker(\alpha))$.

3.C. **The structure sheaf and the category $\mathcal{A}(G)$.** We now turn to localization functors. We introduce terminology so that we can view them as giving sheaves of functions on the space of subgroups.

For each closed connected subgroup K of G we consider the set $U(K)$ of subgroups containing K (which can be identified with the set of subgroups of G/K). We view the collection

$$\mathcal{U} = \{U(K) \mid K \text{ a connected subgroup } \}$$

as the generating set for the U-topology on the set of subgroups of G. We carry the letter U throughout the discussion to distinguish it from a second topology introduced in [9].

A U-sheaf M is a contravariant functor $M : \mathcal{U} \longrightarrow \mathbf{AbGp}$. (The terminology is reasonable since any cover of a set $U(K)$ by sets from $\mathcal{U}$ must involve $U(K)$ itself, so the sheaf condition is automatically satisfied). Thus if K and L are connected with $L \subseteq K$ then $U(L) \supseteq U(K)$ and there is a restriction map $M(U(L)) \longrightarrow M(U(K))$. Note that this is covariant for the inclusion of subgroups and is therefore simply another way of speaking of a localization functor.

We may construct a U-sheaf from the ring $\mathcal{O}_{\mathcal{F}}$.

Definition 3.3. (i) The structure U-sheaf $\mathcal{O}$ is defined by

$$\mathcal{O}(U(K)) = \mathcal{E}_K^{-1}\mathcal{O}_{\mathcal{F}},$$

and the structure maps are the localizations. Thus $\mathcal{O}$ is a U-sheaf of rings, and its ring of global sections is $\mathcal{O}_{\mathcal{F}}$.
(ii) A sheaf of $\mathcal{O}$-modules is a U-sheaf M with the additional structure that $M(U(H))$ is a module over $\mathcal{O}(U(H)) = \mathcal{E}_H^{-1}\mathcal{O}_{\mathcal{F}}$. The restriction maps are required to be module maps: if $L \subseteq K$, the restriction map

$$M(U(L)) \longrightarrow M(U(K)),$$

for the inclusion $U(L) \supseteq U(K)$ is required to be a map of $\mathcal{O}(U(L))$-modules.

We shall be working almost exclusively with sheaves M of $\mathcal{O}$-modules: the standard model for rational G-spectra will be a category of dg sheaves of $\mathcal{O}$-modules with additional structure.

First we restrict attention to modules which are determined by their value on $U(1)$. This is analogous to forming a sheaf over spec(R) from an R-module M: its values over the open set on which x is invertible is $M[1/x]$. We also borrow the well-established and unwieldy terminology from this situation.

Definition 3.4. A *quasi-coherent* U-sheaf (qc U-sheaf) of $\mathcal{O}$-modules is one in which for each connected subgroup K, the restriction map $M(U(1)) \longrightarrow M(U(K))$ is the map inverting the multiplicatively closed set $\mathcal{E}_K$.

Remark 3.5. (i) The structure sheaf $\mathcal{O}$ is quasi-coherent.
(ii) For a quasicoherent sheaf, all values $M(U(H))$ are determined by the value $M(U(1))$.
(iii) The quasi-coherence condition has a major effect. For example, if M is a quasi-coherent module only nonzero on $U(1)$ then $M(U(1))$ is necessarily a torsion module, since it vanishes if we invert $\mathcal{E}_K$ for any non-trivial K.

The second restriction is to sheaves of $\mathcal{O}$-modules which are extended from quotient groups in the following sense.

Definition 3.6. A sheaf M of $\mathcal{O}$-modules is *extended* if we are given a tensor decomposition

$$M(U(K)) = \mathcal{E}_K^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}/K}} \phi^K M$$

where $\phi^K M$ is an $\mathcal{O}_{\mathcal{F}/K}$-module, so that $\{\phi^\bullet M\}$ is an inflation system of $\mathcal{O}_{\mathcal{F}}$-modules. This splitting must be compatible with restriction maps in that if $L \subseteq K$, the restriction is obtained from a map

$$\phi^L M \longrightarrow \mathcal{E}_{K/L}^{-1}\mathcal{O}_{\mathcal{F}/L} \otimes_{\mathcal{O}_{\mathcal{F}/K}} \phi^K M$$

by extension of scalars. A morphism of extended modules is required to arise from a map of inflation systems: if $\theta : M \longrightarrow N$ is a morphism of extended modules, for each K we have a diagram

$$\begin{array}{ccc} M(U(K)) & \stackrel{\theta(U(K))}{\longrightarrow} & N(U(K)) \\ =\downarrow & & \downarrow= \\ \mathcal{E}_K^{-1}\mathcal{O}_{\mathcal{F}} \otimes \phi^K M & \stackrel{1\otimes\phi^K\theta}{\longrightarrow} & \mathcal{E}_K^{-1}\mathcal{O}_{\mathcal{F}} \otimes \phi^K N. \end{array}$$

We write e-$\mathcal{O}$-mod for the category of extended $\mathcal{O}$-modules.

Remark 3.7. (i) The condition on restriction maps makes sense since

$$\mathcal{E}_L^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}/L}} \mathcal{E}_{K/L}^{-1}\mathcal{O}_{\mathcal{F}/K} \otimes_{\mathcal{O}_{\mathcal{F}/K}} (\cdot) = \mathcal{E}_K^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}/K}} (\cdot).$$

The point here is that representations α of G/L with $\alpha^{K/L} = 0$ (whose Euler classes lie in $\mathcal{E}_{K/L}$) map to representations of G with $\alpha^K = 0$ (whose Euler classes lie in $\mathcal{E}_K$) under inflation.
(ii) The structure sheaf $\mathcal{O}$ is extended since

$$\mathcal{E}_K^{-1}\mathcal{O}_{\mathcal{F}} = \mathcal{E}_K^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}/K}} \mathcal{O}_{\mathcal{F}/K}.$$

(iii) The splitting of $M(U(K))$ is specified by the *basing map*

$$\phi^K M \longrightarrow M(U(K))$$

corresponding to the inclusion of the unit.
(iv) The reason for the notation is that $\phi^K M$ is analagous to the value $E_*^{G/K}(\Phi^K X)$ of a cohomology theory on geometric fixed points (see also 3.10).

Remark 3.8. We may therefore think of a quasi-coherent extended U-sheaf M of $\mathcal{O}$-modules as an $\mathcal{O}_{\mathcal{F}}$-module $M(U(1))$ together with additional structure. The additional structure specifies particular "relative trivializations" of $\mathcal{E}_K^{-1}M(U(1))$:

$$\mathcal{E}_K^{-1}M(U(1)) = M(U(K)) = \mathcal{E}_K^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}/K}} \phi^K M.$$

The whole structure is given by $M(U(1))$ together with basing maps $\phi^K M \longrightarrow \mathcal{E}_K^{-1}M(U(1))$ giving the splittings.

Finally, we may introduce the class of sheaves directly relevant to us. The explicit identification of this category is, perhaps, the main achievement of this paper.

Definition 3.9. The *standard abelian category*

$$\mathcal{A} = \mathcal{A}(G) = \text{qce-}\mathcal{O}\text{-mod}$$

is the category of all quasi-coherent extended U-sheaves of $\mathcal{O}$-modules (qce $\mathcal{O}$-modules). We also use the notation

$$\hat{\mathcal{A}} = \hat{\mathcal{A}}(G) = \text{e-}\mathcal{O}\text{-mod}$$

for the category of extended modules with no restriction on structure maps.

It is useful to have an algebraic analogue of the fixed point functor. This is defined on extended $\mathcal{O}$-modules.

Lemma 3.10. *There is a functor*

$$\Phi^L : e\text{-}\mathcal{O}_G\text{-}mod \longrightarrow e\text{-}\mathcal{O}_{G/L}\text{-}mod$$

defined by

$$(\Phi^L M)(U(K/L)) = \mathcal{E}^{-1}_{K/L}\mathcal{O}_{\mathcal{F}/L} \otimes_{\mathcal{O}_{\mathcal{F}/K}} \phi^K M,$$

or equivalently

$$\phi^{K/L}(\Phi^L M) = \phi^K M.$$

This functor takes quasi-coherent modules to quasi-coherent modules. □

Example 3.11. The fixed point functor takes the G-structure sheaf $\mathcal{O}$ to the G/L-structure sheaf: there is an equivalence

$$\Phi^L \mathcal{O}_G = \mathcal{O}_{G/L}$$

of sheaves of modules on the toral chain category for G/L. □

4. A filtration of the standard abelian category $\mathcal{A}(G)$.

In this section we show that any object of the abelian category $\mathcal{A}(G)$ can be built up from objects $f_H(N)$ arising from modules N over the rings $\mathcal{O}_{\mathcal{F}/H}$ for various connected subgroups H. The object $f_H(N)$ is zero on $U(K)$ unless $K \subseteq H$, and it is constant where it is non-zero.

The key to decomposing objects in this way is the fact that all restriction maps $M(U(L)) \longrightarrow M(U(K))$ go in one direction: they increase the dimension of the subgroups. The topological explanation of this phenomenon is just as in [4]. In equivariant stable homotopy one expects to have to deal with restriction maps (decreasing the size of subgroups) and transfers (increasing the size of subgroups). The restriction maps are built into the structure at an early stage. Because we work over the rationals, the Burnside rings of finite groups are products of copies of $\mathbb{Q}$, so that transfer maps for inclusions of finite index can be expressed entirely in terms of idempotents from Burnside rings, so no maps increasing the size of subgroups are necessary. The transfer maps for toral inclusions are zero, and the residual structure comes from the localization theorem.

This filtration is fundamental for calculation, and perhaps the first striking consequence is that the category $\mathcal{A}(G)$ has finite injective dimension. This is the key to the power of $\mathcal{A}(G)$ in the study of G-equivariant cohomology theories.

Subsection 4.A introduces the method for constructing objects of $\mathcal{A}(G)$ from modules, Subsection 4.C shows how arbitrary objects can be constructed from these, and Subsection 5 deduces consequences for homological algebra. In [9, 8.1] we take this further to show the exact injective dimension is the rank of G.

4.A. **Evaluation and extension.** For a chosen connected subgroup K, evaluation gives a functor

$$\mathrm{ev}_K : \mathcal{O}\text{-mod} \longrightarrow \mathcal{O}(U(K))\text{-modules}$$

defined by

$$M \longmapsto M(U(K)).$$

This functor has a right adjoint

$$c_K : \mathcal{O}(U(K))\text{-modules} \longrightarrow \mathcal{O}\text{-mod}$$

given by taking the sheaf constant below K:

$$c_K(N)(U(H)) = \begin{cases} N & \text{if } H \subseteq K \\ 0 & \text{if } H \not\subseteq K \end{cases}$$

The unit of the adjunction

$$\eta : M \longrightarrow c_K \mathrm{ev}_K M$$

is defined to be the restriction $\eta(U(L)) : M(U(L)) \longrightarrow M(U(K))$ if $L \subseteq K$ and is zero otherwise. The counit

$$\epsilon : ev_K c_K N \longrightarrow N$$

is the identity. Thus we have an adjunction

$$\mathrm{ev}_K : \mathcal{O}\text{-mod} \rightleftarrows \mathcal{O}(U(K))\text{-modules} : c_K$$

with the left adjoint on top.

This adjunction obviously restricts to an adjunction between extended $\mathcal{O}$-modules and extended $\mathcal{O}(U(K))$-modules, and if we identify extended $\mathcal{O}(U(K))$-modules with modules for $\mathcal{O}_{\mathcal{F}/K}$, this gives the adjunction

$$\phi^K : \text{e-}\mathcal{O}\text{-mod} \rightleftarrows \mathcal{O}_{\mathcal{F}/K}\text{-modules} : f_K .$$

Explicitly, $f_K(V)$ is constant below K at $\mathcal{E}_K^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}/K}} V$. In other words,

$$f_K(V) = c_K(\mathcal{E}_K^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}/K}} V).$$

A little more care is necessary for quasi-coherent sheaves. Indeed, the U-sheaf $c_K(N)$ will not be quasi-coherent unless (i) $\mathcal{E}_K$ is invertible on N and (ii) $\mathcal{E}_{K'}^{-1}N = 0$ when $K' \not\subseteq K$. We call $\mathcal{O}_{\mathcal{F}}$-modules satisifying (i) *$\mathcal{E}_K$-invertible modules*, and those satisfying (ii) *K-torsion modules*. We call sheaves with $M(U(K')) = 0$ when $K' \not\subseteq K$, *sheaves concentrated below* K. Since quasi-coherent sheaves form a full subcategory, this is the only obstacle, and we have an adjunction

$$\mathrm{ev}_K : \text{qc-}\mathcal{O}\text{-mod-below-}K \rightleftarrows \mathcal{E}_K\text{-inv-}K\text{-torsion-}\mathcal{O}_{\mathcal{F}}\text{-modules} : c_K .$$

Finally, on qce $\mathcal{O}$-modules we combine these to give the adjunction we actually need.

Lemma 4.1. *For any connected subgroup K there is an adjunction*

$$\phi^K : \textit{qce-}\mathcal{O}\textit{-mod-below-}K \rightleftarrows \textit{torsion-}\mathcal{O}_{\mathcal{F}/K}\textit{-modules} : f_K .$$

Furthermore, for any torsion $\mathcal{O}_{\mathcal{F}/K}$-module V and an arbitrary extended module M,

$$\mathrm{Hom}_{\mathcal{O}_{\mathcal{F}/K}}(\phi^K M, V) = \mathrm{Hom}(M, f_K(V)). \quad \square$$

4.B. **Finiteness of quasi-coherent sheaves.** If M is a sheaf of $\mathcal{O}$-modules, then $M(U(K))$ is a module over the ring $\mathcal{O}_{\mathcal{F}/K} = \prod_{\tilde{K}} H^*(BG/\tilde{K})$. This is not a particularly easy ring to work with, and in particular it is not Noetherian. However the quasi-coherence condition on modules means that the relevant module theory is much better behaved. This good behaviour can mostly be traced back to the crucial lemmas proved in this section.

First we should formalize some terminology for $\mathcal{O}$-modules, since there are several notions that could loosely be called 'support', but which should not be confused with the usual use of the word in commutative algebra.

Definition 4.2. (i) If M is a U-sheaf,

$$\mathrm{supp}(M) := \{K \in \mathrm{ConnSub}(G) \mid M(U(K)) \neq 0\}.$$

(ii) If M is a U-sheaf, and $x \in M(U(L))$,

$$\mathrm{supp}_L(x) := \{K \in \mathrm{ConnSub}(G/L) \mid x \text{ has non-zero image in } M(U(K))\}.$$

(iii) If N is a module over $\mathcal{O}_{\mathcal{F}/K}$ and $x \in N$, we say that the *spread* of x is

$$\mathrm{spread}(x) := \{\tilde{K} \in \mathcal{F}/K \mid e_{\tilde{K}} x \neq 0\}.$$

We have finiteness conditions on the support and on the spread of elements. That on support is more elementary.

Lemma 4.3. *Suppose M is a qce-sheaf, L is a connected subgroup of G and $x \in M(U(L))$. If $\mathrm{supp}_L(x)$ consists of subgroups of dimension $\leq s$ then it contains only finitely many subgroups of dimension s.*

Proof: All subgroups considered in this proof are assumed connected and to contain L. The hypothesis states that for each subgroup H of dimension more than s there is a representation $V(H)$ with $V(H)^H = 0$, and $e(V(H))x = 0$. It follows that x maps to zero in $M(U(K))$ whenever $V(H)^K = 0$. Thus if $V(H) = \alpha_1(H) \oplus \cdots \oplus \alpha_{n(H)}(H)$, we see that K can only be in the support of x if $K \subseteq \ker(\alpha_i(H))$ for some i.

For $r = s$ there is nothing to prove, so we suppose $s < r$. We now argue by induction on the codimension c, that if $K \in \mathrm{supp}_L(x)$ there are only finitely many subgroups H of codimension c that contain K. The result for $c = r - s$ is the statement of the lemma. For $c = 0$ we see that K must be contained in $\ker(\alpha_i(G))$ for some i. Now suppose the result is proved in codimension $c < r - s$, and pick one of the finitely many subgroups H of codimension c. Subgroups in the support inside H must lie in one of the subgroups $\ker(\alpha_i(H))$, and hence in one of the finitely many codimension $c+1$ subgroups $H \cap \ker(\alpha_i(H))$. This completes the inductive step. □

The proof of the result on spread uses a result from Section 5, but the result is stated here for ease of reference.

Lemma 4.4. *Suppose M is a qce-sheaf, L is a connected subgroup of G of dimension s and $x \in \phi^L M$. If $1 \otimes x \in \ker(M(U(L)) \longrightarrow M(U(K)))$ for all connected subgroups K containing H of dimension $s+1$ then x has finite spread.*

Proof: Since $\mathcal{E}_L^{-1}\mathcal{O}_{\mathcal{F}}$ is flat over $\mathcal{O}_{\mathcal{F}/L}$ by 5.7, it suffices to consider the special case $L = 1$. The hypothesis states that for each 1-dimensional connected subgroup K, there is a representation $V(K)$ with $V(K)^K = 0$, and $e(V(K))x = 0$.

Now for each finite subgroup F the Euler class $e(\alpha)(F)$ is 1 if $\alpha^F = 0$ and is $c(\alpha) \in H^2(BG/F)$ if α is a representation of G/F. Thus if $V(K) = \alpha_1(K) \oplus \cdots \oplus \alpha_{n(K)}(K)$, the F-coordinate of $e(V(K))$ is 1 unless $F \subseteq \ker(\alpha_i)$ for some i. Hence, writing $\mathcal{F}M$ for the set of finite subgroups of M,

$$\mathrm{spread}(x) \subseteq \mathcal{F}\ker(\alpha_1(K)) \cup \mathcal{F}\ker(\alpha_2(K)) \cup \cdots \cup \mathcal{F}\ker(\alpha_n(K)).$$

More generally, we argue by induction that for $i = 1, 2, 3, \ldots, r$

$$\mathrm{spread}(x) \subseteq \mathcal{F}H_1^i \cup \mathcal{F}H_2^i \cup \cdots \cup \mathcal{F}H_{m(i)}^i$$

for certain subgroups H_j^i of codimension at least i. We have already dealt with the case $i = 1$, and the case $i = r$ shows there are a finite number of subgroups in spread(x). For the inductive step we suppose $i < r$. If all the subgroups already have codimension at least $i + 1$ we can take $H_j^{i+1} = H_j^i$. Otherwise, for each j with H_j^i infinite we can find $K_j \subseteq H_j^i$. The inductive hypothesis together with the spread condition for the K_j shows that we may take the H_k^{i+1} to be intersections of H_j^i and the $\ker(\alpha_s(K_j))$; since $K_j \not\subseteq \ker(\alpha_s(K_j))$, these subgroups are of codimension $\geq i + 1$. This completes the inductive step. □

4.C. *U*-**sheaves are constructed from constant ones.** The category of qce U-sheaves is an abelian category, and we will need to do homological algebra in it. We use the modules constant below some point to import convenient objects into the category of $\mathcal{O}$-modules from categories of modules over suitable rings. These objects suffice to build everything, and the method for proof below gives a practical method of calculation.

Theorem 4.5. *The qce U-sheaves constant below connected subgroups (i.e., the sheaves of the form $f_K(V)$ for some connected subgroup K and some torsion $\mathcal{O}_{\mathcal{F}/K}$-module V) generate the category of all qce U-sheaves using short exact sequences and sums.*

Proof: We say that a U-sheaf is supported on a set of subgroups $\mathcal{K}$ if $M(U(K')) = 0$ when $K' \notin \mathcal{K}$. We argue by finite induction on s that qce sheaves supported on subgroups of dimension $\leq s$ are generated by U-sheaves constant below some point. The induction begins since the statement is obvious with $s = -1$, and the theorem is the case $s = r$.

Suppose then that qce U-sheaves supported on subgroups of dimension $\leq s - 1$ are generated by U-sheaves constant below some point, and that M is a qce U-sheaf supported on subgroups of dimension $\leq s$. For each connected subgroup L of dimension s we note as in Remark 3.5(iii) that $M(U(L))$ is torsion and lift the identity map $M(U(L))$ to a map $M \longrightarrow f_L(M(U(L)))$. Now combine these to a map

$$M \longrightarrow \prod_{\dim(L)=s} f_L(M(U(L))).$$

The product is the termwise product of vector spaces, and therefore not a qce sheaf.

The sheaf M is supported in dimension $\leq s$ so the image of the map satisfies the hypotheses of 4.3, and the map into the product actually maps into the sum. We thus obtain

$$g : M \longrightarrow \bigoplus_{\dim(L)=s} f_L(M(U(L))).$$

The first point is that the sum is a qce U-sheaf since localization and tensor products commute with direct sum.

Next the map g is an isomorphism at $U(H)$ whenever $\dim(H) \geq s$. The kernel and cokernel are then supported on subgroups of dimension $\leq s-1$ and hence constructed from constant sheaves by induction. □

Remark 4.6. If we view a qce-sheaf M as a refinement of the module $M(U(1))$, the construction described above is a refinement of a generalized Cousin complex for $M(U(1))$. For example the first map

$$\eta : M \longrightarrow f_G(M(U(G))$$

evaluated at $U(1)$ is

$$\eta(U(1)) : M(U(1)) \longrightarrow \mathcal{E}_G^{-1}M(U(1)),$$

and the next step

$$\mathrm{cok}(\eta) \longrightarrow \bigoplus_M f_M(\phi^M \mathrm{cok}(\eta)))$$

evaluated at $U(1)$ is

$$\mathrm{cok}(\eta(U(1))) \longrightarrow \bigoplus_M \mathcal{E}_M^{-1}\mathrm{cok}(\eta(U(1))).$$

At each stage the kernel is already understood by induction.

5. Homological algebra of categories of sheaves.

The fact that the category of qce-sheaves has finite injective dimension is fundamental for the convergence of our spectral sequence. The fact that it is small makes the method of calculation practical. The idea is simple: since the f constructions are right adjoints to evaluation we obtain a good supply of injectives, and since the cohomology rings $H^*(BG/K)$ are polynomial rings on at most r generators the injective dimension is r. Implementing this idea is a little more complicated because we need to work with non-Noetherian rings like $\mathcal{O}_{\mathcal{F}}$.

5.A. **Sums of injectives.** One of the convenient things about Noetherian rings is that arbitrary sums of injectives are injective. This is not true for the category of arbitrary modules over $\mathcal{O}_{\mathcal{F}}$. However the qce condition is sufficient to obtain the required property for the categories relevant to us.

Lemma 5.1. *If M is a qce-sheaf and N_L is an $\mathcal{O}_{\mathcal{F}/L}$-module for each connected subgroup L of dimension s then*

$$\mathrm{Hom}(M, \bigoplus_L f_L(N_L)) \xrightarrow{=} \mathrm{Hom}(M, \prod_L f_L(N_L))$$
$$= \prod_L \mathrm{Hom}(M, f_L(N_L)) = \prod_L \mathrm{Hom}(M(U(L)), N_L).$$

The corresponding result holds if L runs through all subgroups, and not just the connected ones.

Proof: First consider the statement with L running through connected subgroups. Any map

$$f : M \longrightarrow \bigoplus_L f_L(N_L).$$

is determined by its behaviour on the sets $U(L)$ because the map

$$\bigoplus_L f_L(N_L) \longrightarrow \prod_L f_L(N_L)$$

of U-sheaves is a termwise monomorphism. Now replace M by its image, to obtain a sheaf supported in dimension $\leq s$. Applying 4.3, any map into the product factors through the sum.

The same proof gives the conclusion for all subgroups, now quoting 4.4. $\square$

This lets us identify a sufficient supply of injective modules.

Corollary 5.2. *If $I_{\tilde{L}}$ is a torsion injective $H^*(BG/\tilde{L})$-module for each subgroup $\tilde{L}$ with identity component L, then $\bigoplus_{\tilde{L}} f_L(I_{\tilde{L}})$ is an injective qce-sheaf.* $\square$

5.B. **The injective dimension of $\mathcal{A}(G)$.** In fact the injective dimension of $\mathcal{A}(G)$ is equal to the rank r of G. This will be proved in [9, 8.1]. For the construction and convergence of the Adams spectral sequence it is enough to show the injective dimension is finite. In fact, the proof below shows that the injective dimension is $\leq 2r$. One expects the method to establish the exact bound, but this would involve delicate arguments to justify the behaviour of the category of modules occurring as $M(U(1))$ for qce-sheaves M.

Theorem 5.3. *The category of qce $\mathcal{O}$-modules has injective dimension $\leq 2r$.*

Proof: We prove by induction on s that any qce-sheaf M supported on subgroups of dimension $\leq s$ is of injective dimension at most $r+s$ ($\mathrm{id}(M) \leq r+s$).

If M is a qce-sheaf supported in dimension 0, then $M = f_1(N)$ and by 5.1 we have a decomposition $N = \bigoplus_F N_F$ with N_F a torsion $H^*(BG/F)$-module. Since each N_F has an injective resolution of length $\leq r$ consisting of torsion modules; indeed, any torsion module embeds in a sum of copies of the injective hull of the residue field, and the cokernel is again a torsion module. We can add these injective modules and apply f_1 to obtain an injective resolution of M.

Now suppose $s \geq 1$, and that the theorem has been proved for sheaves supported in dimension $s-1$. If M is supported in dimension s, we may consider the connected subgroups L of dimension s and the map

$$j : M \longrightarrow E = \bigoplus_{\dim L = s} f_L(\phi^L M).$$

of 4.5. By definition this map is an isomorphism at each subgroup L of dimension s, so that its kernel and cokernel are supported in dimension $\leq s-1$. Thus we have two short exact sequences

$$0 \longrightarrow M' \longrightarrow M \longrightarrow I \longrightarrow 0$$

and

$$0 \longrightarrow I \longrightarrow E \longrightarrow M'' \longrightarrow 0.$$

Since M' and M'' are supported in dimension $\leq s-1$ by construction, they have injective dimension $\leq r+s-1$ by induction. It therefore follows that $\mathrm{id}(M) \leq r+s$ if $\mathrm{id}(I) \leq r+s$, so we may concentrate on the second exact sequence.

Next, $\phi^L M = \bigoplus_{\tilde{L}} M_{\tilde{L}}$, and by 5.2, we see that the injective dimension of E is the maximum of those of the $M_{\tilde{L}}$, and hence $\leq r$ since this is true of $H^*(BG/\tilde{L})$-modules. It follows from the second exact sequence that $\mathrm{id}(I) \leq r+s$. □

5.C. **Realizing enough injectives.** When we come to connections with G-spectra we need to know we can realize enough injectives, and accordingly it is good to have a small list of injectives.

Lemma 5.4. *There are enough injective qce-sheaves of $\mathcal{O}$-modules which are sums of those of the form*

$$I(\tilde{L}) = f_L(H_*(BG/\tilde{L}))$$

where $\tilde{L}$ is a subgroup with identity component L.

Proof: By Theorem 4.5 and 5.2, it suffices to observe that qce-sheaves $f_L(N_{\tilde{L}})$ with $N_{\tilde{L}}$ a torsion $H^*(BG/\tilde{L})$-module have resolutions using modules of the form $f_L(I(\tilde{L}))$. The module $H_*(BG/\tilde{L})$ is the $\mathbb{Q}$-dual of $H^*(BG/\tilde{L})$ and therefore injective over it; indeed, it is the injective hull of the residue field. Any torsion $H^*(BG/\tilde{L})$-module may be embedded in a sum of copies of this, and the quotient is again a torsion module. □

5.D. **Hausdorff modules and flatness.** It is essential in several places to know that to know that the process of extension loses no information, which corresponds to the idea that we can recover all G/K-information from the inflated G-equivariant information by passage to geometric K-fixed points.

First note that the rings $\mathcal{O}_{\mathcal{F}}$ and $\mathcal{O}_{\mathcal{F}/K}$ are products of polynomial rings. Quite generally, if $R = \prod_i R_i$ is a product of commutative rings, and M is an R-module, we write M_i for the summand corresponding to R_i. We say that an R-module M is *Hausdorff* if the map $M \longrightarrow \prod_i M_i$ is a monomorphism. To see this is a restriction, note that $\prod_i M_i / \bigoplus_i M_i$ is not Hausdorff if infinitely many of the M_i are non-zero.

Lemma 5.5. *The inflation map $\mathcal{O}_{\mathcal{F}/K} \longrightarrow \mathcal{O}_{\mathcal{F}}$ and the localization $\mathcal{O}_{\mathcal{F}/K} \longrightarrow \mathcal{E}_K^{-1}\mathcal{O}_{\mathcal{F}}$ are faithfully flat on the category of Hausdorff modules.*

Proof: First, it is clear that individual inflation maps $H^*(BG/\tilde{K}) \longrightarrow H^*(BG/F)$ are free and hence faithfully flat. It follows that $H^*(BG/\tilde{K}) \longrightarrow \prod_{FK=\tilde{K}} H^*(BG/F)$ is faithfully flat. The map $\mathcal{O}_{\mathcal{F}/K} \longrightarrow \mathcal{O}_{\mathcal{F}}$ is a product of such maps.

Quite generally if we have modules M_i faithfully flat over rings R_i, then $\prod_i M_i$ is faithfully flat over $\prod_i R_i$ in the category of Hausdorff modules. For flatness, we note that the product is exact and the map

$$(\prod_i M_i) \otimes N \longrightarrow \prod_i (M_i \otimes_{\prod_i R_i} N)$$

is a monomorphism. For faithful flatness, we note that $\prod_i M_i \otimes_{\prod_i R_i} N$ has $M_i \otimes_{R_i} N$ as a retract, so by faithful flatness, if the former were zero then $e_i N = 0$, and so by the Hausdorff property, N would be zero.

Flatness of the localization follows since localization can be effected by a filtered colimit. Faithfulness uses the special nature of the set of elements inverted. Indeed, the colimit diagram consists of multiplication by $e(W)$ where $W^K = 0$. We claim that if α is a simple representation with $\alpha^K = 0$ then the map $e(\alpha) : \mathcal{O}_{\mathcal{F}} \longrightarrow \mathcal{O}_{\mathcal{F}}$ is a split monomorphism of $\mathcal{O}_{\mathcal{F}/K}$-modules. Indeed, the map splits as a product over finite subgroups F, and it suffices to prove the result for each factor. If $\alpha^F = 0$ the map is the identity, which is certainly split mono. If $\alpha^F \neq 0$ the map is

$$c_1(\alpha) : H^*(BG/F) \longrightarrow H^*(BG/F)$$

viewed as a map of modules over $H^*(BG/FK)$. Indeed, we may choose a splitting $H^*(BG/F) = H^*(BG/FK) \otimes H^*(FK)$, and we may choose polynomial generators for $H^*(FK)$, including $c_1(\alpha)$ amongst them. Using the monomials as an $H^*(BG/FK)$-basis of $H^*(BG/F)$ it is clear that the map is a split monomorphism. □

Lemma 5.6. *If X is a qce module then the modules $\phi^K X$ are all Hausdorff.*

Proof: As in [6, 5.10.1] we see that a submodule of a Hausdorff module is Hausdorff, and that the class of Hausdorff modules is closed under taking sums, products and extensions.

We argue by induction on the rank of G. The case when $G = 1$ is trivial. Now suppose G is of rank $r \geq 1$ and that the result has been proved for smaller rank. In particular if $K \neq 1$, it follows from the result for G/K that $\phi^K X$ is Hausdorff.

Write $M = \phi^1 X = X(U(1))$. The argument that M is Hausdorff is by induction on the dimension of support of X. If X is supported in dimension 0 then $M = \bigoplus_F M_F$ by 4.3, so we may suppose X is supported in dimension $d \geq 1$ and the result has been proved for modules supported in smaller dimension. Now consider the exact sequence

$$0 \longrightarrow X' \longrightarrow X \longrightarrow \bigoplus_{\dim(K)=d} f_K(\phi^K X) \longrightarrow X'' \longrightarrow 0$$

of 5.3, and let $I = \mathrm{im}(X \longrightarrow \bigoplus_{\dim(K)=d} f_K(\phi^K X))$. By induction $X'(U(1))$ is Hausdorff, so the exact sequence

$$0 \longrightarrow X' \longrightarrow X \longrightarrow I \longrightarrow 0$$

shows it suffices to prove $I(U(1))$ is Hausdorff. However I is a subobject of the sum, so it suffices to show $f_K(\phi^K X)(U(1))$ is Hausdorff for all K. In other words, we need to show that if N is a Hausdorff $\mathcal{O}_{\mathcal{F}/K}$-module then $\mathcal{E}_K^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}/K}} N$ is a Hausdorff $\mathcal{O}_{\mathcal{F}}$-module.

For this, we note $\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}/K}} N$ is Hausdorff since the F-th summand is $H^*(BG/F) \otimes_{H^*(BG/KF)} N_{KF}$. Since $\mathcal{E}_K^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}/K}} N$ is a colimit of these Hausdorff modules under split monomorphisms by 5.5, it follows that it is Hausdorff. □

Corollary 5.7. *The inflation map $\mathcal{O}_{\mathcal{F}/K} \longrightarrow \mathcal{O}_{\mathcal{F}}$ and the localization $\mathcal{O}_{\mathcal{F}/K} \longrightarrow \mathcal{E}_K^{-1}\mathcal{O}_{\mathcal{F}}$ are faithfully flat on the category of modules ocurring as $\phi^K M$.* □

Part 3. **The Adams spectral sequence.**

In Part 2 we introduced the algebraic category $\mathcal{A}(G)$, and in Part 3 we provide the connection with G-equivariant cohomology theories by defining the functor $\pi_*^{\mathcal{A}}$ and constructing an Adams spectral sequence based on it.

The basis of the connection is the calculation

$$\mathcal{O}_{\mathcal{F}} = [E\mathcal{F}_+, E\mathcal{F}_+]^G_*$$

of the endomorphism ring of $E\mathcal{F}_+$. This is completed in Section 7. In preparation we begin by understanding the basic building blocks and how they are related to each other.

6. Basic cells.

The familiar generators in topology are the natural cells G/K_+, but when working stably and rationally these break up further using idempotents from the rationalized Burnside ring, and it is more convenient to consider the resulting pieces. In more detail, smashing with G/K_+ gives a ring homomorphism $[S^0, S^0]^K \longrightarrow [G/K_+, G/K_+]^G$, and $[S^0, S^0]^K$ is isomorphic to the rationalized Burnside ring of finite K-sets $A(K)$. Indeed the map

$$\phi : [S^0, S^0]^K \xrightarrow{\cong} \prod_H \mathbb{Q}$$

with Hth component being given by the degree of geometric H-fixed points is a ring isomorphism.

Definition 6.1. The *basic cell* for the closed subgroup K is defined by

$$\sigma^0_K = e_{K/K_1} G/K_+,$$

where $e_{K/K_1} \in A(K/K_1)$ is the primitive idempotent in the Burnside ring corresponding to the group K/K_1 of components of K.

The usefulness of the basic cells is that they provide decompositions of all the natural cells.

Lemma 6.2. *Suppose $\tilde{K}$ is a subgroup with identity component K_1. There is a decomposition*

$$G/\tilde{K}_+ \simeq \bigvee_{K_1 \subseteq K \subseteq \tilde{K}} \sigma^0_K$$

where the splitting is indexed by subgroups K/K_1 of the group $\tilde{K}/K_1$ of components of $\tilde{K}$.

Proof: We follow the pattern of [6, 2.1.5]. It suffices to show that if $K_1 \subseteq K \subseteq \tilde{K}$ then $G_+ \wedge_K e_K S^0 = \sigma^0_K \simeq G_+ \wedge_{\tilde{K}} e_K S^0$. Indeed, we need only show that $G_+ \wedge_{\tilde{K}} e_K \widetilde{\tilde{K}/K}$ is contractible where $\widetilde{\tilde{K}/K} = \mathrm{cofibre}(\tilde{K}/K_+ \longrightarrow S^0)$.

We suppose $G = G' \times G''$ with $\tilde{K} \subseteq G'$ a product of inclusions of a cyclic group in a circle. It suffices to prove the analogous result with G replaced by G'. The analogue of [6, 2.1.4] replaces a single cofibre sequence by $r' = \mathrm{rank}(G')$ of them. For each of the cyclic factors we apply the method of [6, 2.1.5] to the permutation representation of $\tilde{K}/K$. $\square$

Lemma 6.3. *Maps between basic cells in degree 0 are as follows*

$$[\sigma^0_K, \sigma^0_L]^G_0 = \begin{cases} \mathbb{Q} & \text{if } K \text{ is cotoral in } L \\ 0 & \text{otherwise} \end{cases}$$

Proof: We need only apply idempotents to the corresponding statements with natural cells. Indeed $[G/K_+, \sigma^0_L]^G = [S^0, \sigma^0_L]^K$. This is zero unless L is of finite index in K. If L is of finite index the idempotents e_K and e_L are orthogonal unless $K = L$. Finally, if $K = L$ we note that G/L_+ is L-equivariantly obtained from S^0 by attaching cells of dimension ≥ 1, and hence the desired group is a quotient of $e_K[S^0, S^0]^K = \mathbb{Q}$. It is non-trivial since σ^0_K is not contractible. □

Lemma 6.4. *If F is finite, the endomorphism ring of σ^0_F is exterior on r generators,*

$$[\sigma^0_F, \sigma^0_F]^G_* = \Lambda(H_1(G/F)).$$

Proof: Additively the calculation is correct since $[\sigma^0_F, \sigma^0_F]^G_* = [G/F_+, \sigma^0_F]^G_* = [S^0, G/F_+]^F_*$ and G/F_+ is F-fixed and a torus with added basepoint. The ring structure may be seen by passing to non-equivariant homology

$$[\sigma^0_F, \sigma^0_F]^G_* \longrightarrow \mathrm{Hom}(H_*(\sigma^0_F), H_*(\sigma^0_F)).$$

This is a ring map and, since $H_*(\sigma^0_F) \cong H_*(G/F_+)$, the codomain is exterior. It remains to note that this is surjective in degree 1. This in turn follows from the rank 1 case by the Künneth theorem. The rank 1 case is clear since the degree 1 map is tautologously detected in F-equivariant homotopy and hence in homology. □

Finally we record the Whitehead Theorem for spectra with stable isotropy only at F.

Lemma 6.5. *If F is finite, and X is a spectrum so that*

1. *X has stable isotropy only at F and*
2. *$[\sigma^0_F, X]^G_* = 0$*

then X is contractible.

Proof: By (1) we have $\Phi^K X$ trivial unless K is a subgroup of F. It therefore suffices to show that $[G/K_+, X]^G_* = 0$ if $K \subseteq F$. However, G/K_+ splits as a wedge of basic cells for finite subgroups. Since X only has isotropy at F, the only possible contribution is from the summand $[\sigma^0_F, X]^G_*$, and this is zero by hypothesis. □

7. Endomorphisms of injective spectra.

The basis for the correspondence between algebra and topology is the universal space $E\mathcal{F}_+$ for the collection $\mathcal{F}$ of finite subgroups of G. This plays a central role because its endomorphism ring is so well behaved: the simplicity we see here will have even more power in [11].

We follow the strategy of [6], adapted to account for the fact that the exterior algebra $H_*(G)$ and the polynomial algebra $H^*(BG)$ now have r generators, rather than the single generator in the case of the circle.

First it is convenient to introduce injective counterparts of the basic cells.

Definition 7.1. For any subgroup K we define the G-space $E\langle K\rangle$ by

$$E\langle K\rangle = \text{cofibre}(E[\subset K]_+ \longrightarrow E[\subseteq K]_+).$$

Example 7.2. (i) If $K = 1$ we have $E\langle 1\rangle = EG_+$.
(ii) If $K = G$ we have $E\langle G\rangle = \tilde{E}\mathcal{P}$ where $\mathcal{P}$ is the family of proper subgroups of G.

Between them these give the general picture.

Lemma 7.3. *If K is a subgroup with identity component K_1, then there is an equivalence*

$$\Phi^{K_1} E\langle K\rangle \simeq E\langle K/K_1\rangle$$

of G/K_1-spaces and an equivalence

$$E\langle K\rangle \simeq S^{\infty V(K_1)} \wedge E\langle K/K_1\rangle$$

of G-spaces.

Proof: For any family $\mathcal{H}$ of subgroups and any subgroup L

$$\Phi^L E\mathcal{H}_+ \simeq E\mathcal{H}/L_+$$

where $\mathcal{H}/L$ is the family of subgroups of G/L which are images of those of $\mathcal{H}$. This gives the first statement.

The second statement follows: the only group with non-trivial geometric fixed points on either side is K_1, and for K_1 the equivalence is the first statement. □

We are now ready to identify homotopy endomorphism rings.

Theorem 7.4. *The homotopy endomorphism ring of $E\mathcal{F}_+$ is given by*

$$[E\mathcal{F}_+, E\mathcal{F}_+]^G_* = \mathcal{O}_{\mathcal{F}} = \prod_{F\in\mathcal{F}} H^*(BG/F).$$

Proof: First, as in [3] we may use idempotents to split $E\mathcal{F}_+$:

$$E\mathcal{F}_+ \simeq \bigvee_{F\in\mathcal{F}} E\langle F\rangle.$$

It therefore suffices to prove the corresponding result, Theorem 7.5, about the summands. □

Theorem 7.5. *The homotopy endomorphism ring of $E\langle F\rangle$ is given by*

$$[E\langle F\rangle, E\langle F\rangle]^G_* = H^*(BG/F).$$

The first tool is a characterization of $E\langle F\rangle$.

Proposition 7.6. *If F is finite, the spaces $E\langle F\rangle$ are characterized by*

(1) *$E\langle F\rangle$ has stable isotropy only at F and*
(2) *$[\sigma^0_F, E\langle F\rangle]^G_* = \mathbb{Q}$.*

Proof: First note that $[\sigma_F^0, E\langle F\rangle]^G_* = \mathbb{Q}$. Now we proceed by cellular approximation to construct a map $X \longrightarrow E\langle F\rangle$, where X is constructed from cells σ_F^0 which is an isomorphism of $[\sigma_F^0, \cdot]^G_*$. This is an equivalence by the Whitehead Theorem 6.5. □

We may now identify the endomorphism ring of $E\langle F\rangle$.

Proof of 7.5: Note that $[E\langle F\rangle, E\langle F\rangle]^G_* = [E\langle F\rangle, S^0]^G_*$ so the result will follow additively if we can construct $E\langle F\rangle$ with basic cells in even degrees corresponding to the monomials in $H^*(BG/F)$. The proof is by killing homotopy groups.

In the proof of 7.6 we noted that $E\langle F\rangle$ can be constructed using the basic cell σ_F^0. We repeat the proof, but this time keep track of the cells. By 6.4, the endomorphism ring of σ_F^0 is exterior on r generators. Indeed, let

$$\mathbb{Q} \longleftarrow P_0 \longleftarrow P_1 \longleftarrow P_2 \longleftarrow \cdots$$

be the standard Koszul resolution of $\mathbb{Q}$ by free $\Lambda H_1(G/F)$-modules. Thus

$$P = \Lambda(H_1 G/F)[c_1, c_2, \ldots, c_r].$$

Note that the kernel of each map $P_n \longrightarrow P_{n-1}$ is generated by its bottom degree elements and these are in bijective correspondence with monomials of degree n.

We argue inductively that we may construct (1) a $2n$-dimensional complex $X^{(2n)}$ with basic cells in bijective correspondence with monomials of degree $\leq n$ in the $c_1, c_2, \ldots, c_r$ so that its cellular chain complex is the first n stages of the Koszul resolution and (2) a map $X^{(2n)} \longrightarrow E\langle F\rangle$ which is $2n$-connected. This is certainly true for $r = 0$, so we need only describe the inductive step. However, by construction the bottom degree homotopy generates nth syzygy in the Koszul resolution, so there is no obstruction.

It remains to comment on the ring structure. Consider the cellular filtration, and the resulting spectral sequence for $[S^0, \cdot]^G_*$. We obtain a ring map

$$[E\langle F\rangle, E\langle F\rangle]^G_* \longrightarrow \mathrm{Hom}(H^*(BG/F), H^*(BG/F)).$$

Each generator $c_i \in H^2(BG/F)$ corresponds to a map of resolutions, and we may realize this by a map $E\langle F\rangle \longrightarrow \Sigma^2 E\langle F\rangle$. It follows that the ring map is surjective. By the additive result, it is an isomorphism. □

We also need to know this identification is natural for quotient maps.

Now there is a natural map $E\mathcal{F}_+ \longrightarrow E\mathcal{F}/K_+$ of G-spaces, since every finite subgroup of G has finite image in G/K. Viewing this is a map of spectra and dualizing, we obtain a map $DE\mathcal{F}/K_+ \longrightarrow DE\mathcal{F}_+$. Combined with the inflation map $[S^0, DE\mathcal{F}/K_+]^{G/K}_* \longrightarrow [S^0, DE\mathcal{F}/K_+]^G_*$, and using 7.4 for G and G/K, we obtain a ring homomorphism $\mathcal{O}_{\mathcal{F}/K} \longrightarrow \mathcal{O}_{\mathcal{F}}$.

Lemma 7.7. *The geometrically induced ring homomorphism coincides with the map $q^* : \mathcal{O}_{\mathcal{F}/K} \longrightarrow \mathcal{O}_{\mathcal{F}}$ described in Subsection 3.A, which is the product of the ring homomorphisms*

$$q^*_{\tilde{K}} : \mathcal{O}_{\tilde{K}} \longrightarrow \mathcal{O}_{q_*^{-1}(\tilde{K})} = \prod_{q_*(F) = \tilde{K}/K} H^*(BG/F)$$

where $q_* : \mathcal{F} \longrightarrow \mathcal{F}/K$ is reduction mod K, and the components of $q^*_{\tilde{K}}$ are induced by the quotient maps $G/F \longrightarrow G/\tilde{K}$.

Proof: We have the splitting $E\mathcal{F}_+ \simeq \bigvee_{F\in\mathcal{F}} E\langle F\rangle$ of rational G-spectra [3]. Similarly the stable rational G/K-splitting $E\mathcal{F}/K_+ \simeq \bigvee_{\tilde{K}/K\in\mathcal{F}/K} E\langle \tilde{K}/K\rangle$ may be inflated to a G-splitting. From fixed points one sees that the map $E\mathcal{F}_+ \longrightarrow E\mathcal{F}/K_+$ respects the splitting in the sense that $E\langle F\rangle$ maps trivially to $E\langle \tilde{K}/K\rangle$ unless $q(F) = \tilde{K}/K$. Since duality takes sums to products, 7.5 completes the proof. □

8. Topology of Euler classes.

The next ingredient is to show that the inclusions $S^0 \longrightarrow S^V$ induce suitable Euler classes.

The relevant input from topology comes from the Thom isomorphism for an individual stalk. We once again use the basic injectives 7.1.

Lemma 8.1. *For any finite group F there is an equivalence*

$$S^V \wedge E\langle F\rangle \simeq S^{|V^F|} \wedge E\langle F\rangle.$$

Proof: The cofibre of the map $S^{V^F} \longrightarrow S^V$ is built from cells with isotropy not containing F. It is therefore contractible when smashed with $E\langle F\rangle$. We may thus suppose V is F-fixed.

Now $E\langle F\rangle$ may be built from basic cells σ^0_F. Since

$$G/F_+ \wedge S^{V^F} \simeq G_+ \wedge_F S^{|V^F|} \simeq G/F_+ \wedge S^{|V^F|},$$

we find that

$$\sigma^0_F \wedge S^{V^F} \simeq \sigma^0_F \wedge S^{|V^F|}.$$

Accordingly, $E\langle F\rangle \wedge S^{V^F}$ is also built from cells σ^0_F and

$$[\sigma^0_F, E\langle F\rangle \wedge S^{V^F}]^G_* = [\sigma^0_F, E\langle F\rangle \wedge S^{|V^F|}]^G_* = \Sigma^{|V^F|}\mathbb{Q}.$$

Thus the result follows from 7.6. □

Remark 8.2. Note that the proof displays a specific equivalence on the bottom cell, and hence determines the homotopy class of the equivalence.

As usual, the Thom isomorphism gives rise to an Euler class.

Definition 8.3. The F Euler class $c(V)(F)$ of a representation V is the map

$$S^0 \wedge E\langle F\rangle \longrightarrow S^V \wedge E\langle F\rangle \simeq S^{|V^F|} \wedge E\langle F\rangle.$$

We may identify these Euler classes in familiar terms.

Lemma 8.4. *Under the identification $[E\langle F\rangle, E\langle F\rangle]^G_* = H^*(BG/F)$, the Euler class $c(V)(F)$ is the ordinary cohomology Euler class $c_H(V^F)$.*

Proof: Since both Euler classes take sums of representations to products, it suffices to consider a 1-dimensional representation V. If $V^F = 0$, both Euler classes are 1. If V is fixed by F, then V is a faithful representation of G/K for some $(r-1)$-dimensional subgroup K containing F. Both maps are given by multiplication by a degree 2 class.

It therefore suffices to consider the case of the circle and the representation z^n. The standard generator is the first Euler class $c_H(z)$ and the additive formal group shows $c_H(z^n) =$

$nc_H(z)$. On the other hand, the identification of $G_+ \wedge S^{z^n} \simeq G_+ \wedge S^2$ lets $t(g \wedge x) = tg \wedge x$ in $G_+ \wedge S^2$ correspond to $t(g \wedge x) = tg \wedge t^n x$ in $G_+ \wedge S^{z^n}$, which is a map of degree n. □

In view of the splitting theorem $E\mathcal{F}_+ \simeq \bigvee_{F \in \mathcal{F}} E\langle F \rangle$ we obtain a general Thom isomorphism.

Corollary 8.5. *For any virtual complex representation V and associated dimension function $v : \mathcal{F} \longrightarrow \mathbb{Z}$ defined by $v(F) = \dim_{\mathbb{R}}(V^F)$, there are equivalences*

$$S^V \wedge E\mathcal{F}_+ \simeq \bigvee_F S^{v(F)} \wedge E\langle F \rangle,$$

and

$$S^V \wedge DE\mathcal{F}_+ \simeq \prod_F S^{v(F)} \wedge DE\langle F \rangle.$$

We may now define the global Euler class.

Definition 8.6. The Euler class of a complex representation V is

$$S^0 \wedge E\mathcal{F}_+ \longrightarrow S^V \wedge E\mathcal{F}_+ \simeq \bigvee_F S^{v(F)} \wedge E\langle F \rangle,$$

as a non-homogeneous element of $\mathcal{O}_{\mathcal{F}}$.

Corollary 8.7. *The Euler class, viewed as an element of $\mathcal{O}_{\mathcal{F}}$ has Fth component.*

$$c(V)(F) = c_H(V^F) \in H^*(BG/F). \quad \square$$

9. Sheaves from spectra.

Now that we understand the homotopy endomorphism ring of $E\mathcal{F}_+$ we may forge the link with algebra: since $[E\mathcal{F}_+, E\mathcal{F}_+]^G_* = \mathcal{O}_{\mathcal{F}}$ by 7.4, any spectrum $X \wedge DE\mathcal{F}_+$ has homotopy groups which are $\mathcal{O}_{\mathcal{F}}$-modules. In this section we give the proof that $\pi^{\mathcal{A}}_*$ takes values in $\mathcal{A}(G)$ (stated as 1.5).

From the definition of Euler classes we see that $\pi^{\mathcal{A}}_*(X)$ is quasi-coherent.

Proposition 9.1. *For any G-spectrum X the object $\pi^{\mathcal{A}}_*(X)$ is quasi-coherent in the sense that for any connected subgroup K,*

$$\pi^{\mathcal{A}}_*(X)(U(K)) = \mathcal{E}_K^{-1}\pi^{\mathcal{A}}_*(X)(U(1)).$$

Proof: We combine the definition of $\pi^{\mathcal{A}}_*$ with that of Euler classes to obtain

$$\pi^{\mathcal{A}}_*(X)(U(K)) = \pi^G_*(X \wedge DE\mathcal{F}_+ \wedge S^{\infty V(K)}) = \mathcal{E}_K^{-1}\pi^G_*(X \wedge DE\mathcal{F}_+) = \mathcal{E}_K^{-1}\pi^{\mathcal{A}}_*(X)(U(1)).$$

□

We may now complete the proof that S^0 corresponds to the structure sheaf $\mathcal{O}$.

Proof of Theorem 1.4: Recall that

$$\pi^{\mathcal{A}}_*(X)(U(K)) = \pi^G_*(DE\mathcal{F}_+ \wedge S^{\infty V(K)} \wedge X).$$

Taking $K = 1$, we see that by 7.4, $\pi^{\mathcal{A}}_*(S^0)(U(1)) = \mathcal{O}_{\mathcal{F}}$. By 9.1

$$\pi^{\mathcal{A}}_*(S^0)(U(K)) = \mathcal{E}_K^{-1}\mathcal{O}_{\mathcal{F}} = \mathcal{O}(U(K)).$$

□

Lemma 9.2. *The quasi-coherent U-sheaf $\pi^{\mathcal{A}}_*(X)$ of $\mathcal{O}$-modules is extended. In fact, the value at $U(K)$ splits with*

$$\phi^K \pi^{\mathcal{A}}_*(X) = \pi^{G/K}_*(\Phi^K X \wedge DE\mathcal{F}/K_+)$$

since

$$\pi^G_*(X \wedge DE\mathcal{F}_+ \wedge S^{\infty V(K)}) = \mathcal{E}_K^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}/K}} \pi^{G/K}_*(\Phi^K X \wedge DE\mathcal{F}/K_+)$$

Proof: There is a natural transformation arising from

$$\mathrm{inf}^G_{G/K}(\Phi^K X \wedge DE\mathcal{F}/K_+) \longrightarrow X \wedge DE\mathcal{F}_+ \wedge S^{\infty V(K)}.$$

This gives a natural transformation of homology theories of X, so we need only check it is an isomorphism for various cells $X = G/H_+$. If $X = S^0 = G/G_+$ the map is an isomorphism by definition. The general case follows by the *Rep*(G)-isomorphism argument (Theorem 11.2) since we have Thom isomorphisms on both sides. □

10. Adams spectral sequences.

It is clear that $\pi^{\mathcal{A}}_*$ is functorial and exact, and therefore by 1.5 it defines a homology functor

$$\pi^{\mathcal{A}}_* : G\text{-}\mathbf{spectra} \longrightarrow \mathcal{A}(G).$$

with values in the abelian category $\mathcal{A}(G)$ with injective dimension r.

Theorem 10.1. *The homology theory $\pi^{\mathcal{A}}_*$ gives a convergent Adams spectral sequence which collapses at E_{r+1}.*

We apply the usual method for constructing an Adams spectral sequence based on a homology theory H_* on a category $\mathbb{C}$ with values in an abelian category $\mathcal{A}$ (in our case $\mathbb{C}$ is the category of rational G-spectra, $\mathcal{A} = \mathcal{A}(G)$ and $H_* = \pi^{\mathcal{A}}_*$). It may be helpful to summarize the process: to construct an Adams spectral sequence for calculating $[X, Y]$ we proceed as follows.

Step 0: Take an injective resolution

$$0 \longrightarrow H_*(Y) \longrightarrow I_0 \longrightarrow I_1 \longrightarrow I_2 \longrightarrow \cdots$$

in $\mathcal{A}$.

Step 1: Show that enough injectives I of $\mathcal{A}$ (including the I_j) can be realized by objects $\mathbb{I}$ of $\mathbb{C}$ in the sense that $H_*(\mathbb{I}) \cong I$.

Step 2: Show that the injective case of the spectral sequence is correct in that homology gives an isomorphism

$$[X, Y] \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{A}}(H_*(X), H_*(Y))$$

if Y is one of the injectives $\mathbb{I}$ used in Step 1.

Step 3: Now construct the Adams tower

$$\begin{array}{ccc} \vdots & & \\ \cdot & & \\ \downarrow & & \\ Y_2 & \longrightarrow & \Sigma^{-2}\mathbb{I}_2 \\ \downarrow & & \\ Y_1 & \longrightarrow & \Sigma^{-1}\mathbb{I}_1 \\ \downarrow & & \\ Y_0 & \longrightarrow & \Sigma^{0}\mathbb{I}_0 \end{array}$$

over $Y = Y_0$ from the resolution. This is a formality from Step 2. We work up the tower, at each stage defining Y_{j+1} to be the fibre of $Y_j \longrightarrow \Sigma^{-j}\mathbb{I}_j$, and noting that $H_*(Y_{j+1})$ is the $(j+1)$st syzygy of $H_*(Y)$.
Step 4: Apply $[X, \cdot]$ to the tower. By the injective case (Step 2), we identify the E_1 term with the complex $\mathrm{Hom}^*_{\mathcal{A}}(H_*(X), I_\bullet)$ and the E_2 term with $\mathrm{Ext}^{*,*}_{\mathcal{A}}(H_*(X), H_*(Y))$.
Step 5a: If the injective resolution is infinite, the first step of convergence is to show that $H_*(\underset{\leftarrow\ j}{\mathrm{holim}}\ Y_j)$ is calculated using a Milnor exact sequence from the inverse system $\{H_*(Y_j)\}_j$, and hence that $H_*(\underset{\leftarrow\ j}{\mathrm{holim}}\ Y_j) = 0$. In our case, this is automatic, since the resolution is finite.
Step 5b: Deduce convergence from Step 5a by showing $\underset{\leftarrow\ j}{\mathrm{holim}}\ Y_j \simeq *$. In other words we must show that $H_*(\cdot)$ detects isomorphisms in the sense that $H_*(Z) = 0$ implies $Z \simeq *$. In general, one needs to require that $\mathbb{C}$ is a category of appropriately complete objects for this to be true. This establishes conditional convergence. If $\mathcal{A}$ has finite injective dimension, finite convergence is immediate.

In our case Step 0 follows from Theorem 5.3, and Lemma 5.4 shows only a small list of injectives is required. Steps 3 and 4 are formalities, and Step 5a is automatic since $\mathcal{A}$ has finite injective dimension (5.3). This leaves us to complete Steps 1, 2 and 5b.

For Step 1, we use the basic injective G-spectrum $E\langle K\rangle$ of 7.1 which has stable isotropy only at the subgroup K. The essential property is that this realizes the basic injectives $I(K)$ in $\mathcal{A}(G)$.

Lemma 10.2. *If K is of codimension c we have*

$$I(K) = f_{K_1}(\Sigma^c H_*(BG/K)) = \pi^{\mathcal{A}}_*(E\langle F\rangle)$$

and hence by 5.4 there are enough realizable injectives.

Proof: First, by 7.3, we have

$$E\langle K\rangle = S^{\infty V(K_1)} \wedge E\langle K/K_1\rangle,$$

so that

$$\pi^G_*(DE\mathcal{F}_+ \wedge E\langle K\rangle) = \mathcal{E}^{-1}_{K_1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}/K}} \pi^{G/K_1}_*(DE(\mathcal{F}/K_1)_+ \wedge E\langle K/K_1\rangle).$$

The result therefore follows from the special case in which $K = F$ is finite and $c = r$.

Now, since $S^0 \longrightarrow DE\mathcal{F}_+$ is an F-equivalence, $DE\mathcal{F}_+ \wedge E\langle F\rangle \simeq E\langle F\rangle$ and therefore

$$\pi_*^G(DE\mathcal{F}_+ \wedge E\langle F\rangle) = \Sigma^r H_*(BG/F).$$

Since this is a torsion module

$$\pi_*^{\mathcal{A}}(E\mathcal{F}_+) = f_1(\Sigma^r H_*(BG/F)).$$

□

For Step 2 we prove the injective case of the Adams spectral sequence.

Lemma 10.3. *For any G-spectrum X, application of $\pi_*^{\mathcal{A}}$ induces an isomorphism*

$$[X, E\langle K\rangle]^G \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{A}}(\pi_*^{\mathcal{A}}(X), \pi_*^{\mathcal{A}}(E\langle K\rangle)).$$

Proof: Let $N = \pi_*^{\mathcal{A}}(X)$, and argue by induction on the dimension of G.

For $E\langle K\rangle$ we combine the following diagram

$$\begin{array}{ccc}
[X, E\langle K\rangle]^G & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(N, f_K(\Sigma^c H_*(BG/K)) \\
\downarrow = & & = \downarrow \\
[\Phi^K X, EG/K_+]^{G/K} & \longrightarrow & \mathrm{Hom}_{H^*(BG/K)}(\phi^K N, \Sigma^c H_*(BG/K))
\end{array}$$

with a result for G/K to show the bottom horizontal is an isomorphism.

For notational simplicity we treat the case $K = 1$, where we are left to show

$$\pi_*^G : [X, EG_+]^G \xrightarrow{\cong} \mathrm{Hom}_{H^*(BG)}(\pi_*^G(X \wedge DEG_+), \Sigma^r H_*(BG)).$$

It is easy to see the groups are isomorphic for $X = S^0$. Passage to homology is injective because $\pi_*^G(S^0) \longrightarrow \pi_*^G(S^0 \wedge DEG_+)$ is a monomorphism in degree 0. Since π_*^G compares rational vector spaces of equal finite dimension when $X = S^0$, it is an isomorphism. There are Thom isomorphisms in algebra and topology, so it follows that passage to homology is an isomorphism for $X = S^V$ for any complex representation V. By the *Rep*(G)-isomorphism argument (Theorem 11.2) it is an isomorphism for $X = G/K_+$ for any subgroup K and hence in general. □

Finally, for Step 5b we prove the universal Whitehead Theorem.

Lemma 10.4. *The functor $\pi_*^{\mathcal{A}}$ detects isomorphisms in the sense that if $f : Y \longrightarrow Z$ is a map of G spectra inducing an isomorphism $f_* : \pi_*^{\mathcal{A}}(Y) \longrightarrow \pi_*^{\mathcal{A}}(Z)$ then f is an equivalence.*

Proof: Since $\pi_*^{\mathcal{A}}$ is exact, it suffices to prove that if $\pi_*^{\mathcal{A}}(X) = 0$ then $X \simeq *$. We argue by induction on the dimension of G.

Suppose $\pi_*^{\mathcal{A}}(X) = 0$. From the geometric fixed point Whitehead theorem (see [2] or deduce it from the Lewis-May fixed point Whitehead Theorem [12]) it suffices to show that $\Phi^K X$ is non-equivariantly contractible for all K. Since $\phi^K \pi_*^{\mathcal{A}}(X) = \pi_*^{G/K}(\Phi^K X)$, and $\mathcal{E}_K^{-1}\mathcal{O}_{\mathcal{F}}$ is faithfully flat over $\mathcal{O}_{\mathcal{F}/K}$ by 5.7, the result follows by induction from the G/K-equivariant result provided $K \neq 1$.

It remains to deduce $X \wedge EG_+$ is contractible. Since we have Thom isomorphisms we note that by the *Rep*(G)-isomorphism argument (Theorem 11.2), it suffices to show that

$\pi_*^G(X \wedge EG_+) = 0$. Now $E\mathcal{F}_+ \longrightarrow S^0$ is a non-equivariant equivalence so that $DE\mathcal{F}_+ \wedge EG_+ \simeq DS^0 \wedge EG_+ = EG_+$, so that it is enough to show $\pi_*^G(X \wedge DE\mathcal{F}_+ \wedge EG_+) = 0$.

However if Y has Thom isomorphisms and $\mathbb{C}$ is a one dimensional representation then the cofibre sequence

$$Y \wedge S(\infty\mathbb{C})_+ \longrightarrow Y \longrightarrow Y \wedge S^{\infty\mathbb{C}}$$

shows that if $\pi_*^G(Y) = 0$ then also $\pi_*^G(Y \wedge S(\infty\mathbb{C})_+) = 0$. Writing $EG_+ = S(\infty\mathbb{C}_1)_+ \wedge \ldots \wedge S(\infty\mathbb{C}_r)_+$ we reach the desired conclusion in r steps. □

11. The *Rep*(*G*)-isomorphism argument.

The present section records a method that is useful rather generally in equivariant topology. It has nothing to do with the fact that we are working rationally.

When trying to establish an object is contractible or a map is an equivalence we want to use the most convenient test objects. The Whitehead Theorem says it suffices to use the set $\{G/H_+ \mid H \subseteq G\}$: if $[G/H_+, X]_*^G = \pi_*^H(X) = 0$ for all H then X is contractible.

Definition 11.1. A G-spectrum is *Rep*(G)-contractible if $[S^V, X]_*^G = 0$ for all complex representations V.

It is often easy to see $\pi_*^G(X) = 0$. If we happen to have Thom isomorphisms $S^V \wedge X \simeq S^{|V|} \wedge X$ for complex representations V this shows that X is *Rep*(G)-contractible. It does not necessarily follow that X is contractible even if X is rational and G is abelian (for example if G is cyclic of order 3 and X is a Moore spectrum for a two dimensional simple representation [2]) but it is useful to have a sufficient condition.

Theorem 11.2. *Suppose G is an abelian compact Lie group. If X is a Rep(G) contractible G-spectrum and G/H acts trivially on $\pi_*^H(S^{-V} \wedge X)$ for all H and all desuspensions of X then X is contractible. If G is a torus, the condition of trivial action may be omitted.*

Proof: We argue by induction on the size of G: since compact Lie groups satisfy the descending chain condition on subgroups we can assume the result is true for all proper subgroups. We know $\pi_*^G(X) = 0$ by hypothesis, so by the Whitehead Theorem it suffices to show that $\pi_*^K(X) = 0$ for all proper subgroups. Now any proper subgroup K lies in a subgroup H with G/H a subgroup of the circle. It therefore suffices by induction to establish that X is *Rep*(H)-contractible. Since $H \subseteq G$, any trivial action condition will certainly be inherited by subgroups.

If G/H is a circle, we use the cofibre sequence $G/H_+ \longrightarrow S^0 \longrightarrow S^{V(H)}$ where $V(H)$ has kernel H. We conclude that $[G/H_+, X]_*^G = 0$, and more generally, by smashing with S^V, that $[S^V, X]_*^H = 0$ for any representation V of G. Since G is abelian, every representation of H extends to one of G, and so X is *Rep*(H)-contractible, and hence we conclude X is H-contractible by induction.

If G/H is a finite cyclic group we choose a faithful representation $W(H)$ of G/H and use the cofibre sequence $S(W(H))_+ \longrightarrow S^0 \longrightarrow S^{W(H)}$ and the stable cofibre sequence

$$G/H_+ \xrightarrow{1-g} G/H_+ \longrightarrow S(W(H))_+,$$

where g is a generator of G/H_+. The first shows that $[S(W(H))_+, X]_*^G = 0$, and the second shows that $1 - g$ gives an isomorphism of $[G/H_+, X]_*^G$. By the trivial action condition we conclude $[G/H_+, X]^G_* = 0$, and more generally $[S^V, X]^H_* = 0$. $\square$

Remark 11.3. (i) It suffices to assume that G acts unipotently on $\pi^H_*(X)$ for all H. This is useful for p-groups in characteristic p.
(ii) Variants on this theorem are useful in other contexts. For instance any nilpotent or supersoluble finite group has maximal subgroups which are normal with cyclic quotient. However, not every representation of a maximal subgroup extends to one for G, so additional hypotheses are necessary.
(iii) If we admit real representations, then no trivial action condition is necessary for subgroups of index 2 since the mapping cone of $G/H_+ \longrightarrow S^0$ is S^V for a real representation V.

12. A characterization of basic cells.

We show that basic cells are characterized by their homotopy. Apart from illustrating the use of the Adams spectral sequence, this result is needed in [11].

Theorem 12.1. *If $\pi^{\mathcal{A}}_*(X) \cong \pi^{\mathcal{A}}_*(\sigma^0_H)$ then $X \simeq \sigma^0_H$.*

Proof: We apply the Adams spectral sequence to calculate $[\sigma^0_H, X]^G_0$. In the rest of the section we will show

$$\mathrm{Ext}^{s,t}_{\mathcal{A}}(\pi^{\mathcal{A}}_*(\sigma^0_H), \pi^{\mathcal{A}}_*(\sigma^0_H)) = 0 \text{ for } t - s < s$$

(and hence in particular for $t - s < 0$) so that the isomorphism $\pi^{\mathcal{A}}_*(\sigma^0_H) \xrightarrow{\cong} \pi^{\mathcal{A}}_*(X)$ lifts to a map $\sigma^0_H \longrightarrow X$. Since $\pi^{\mathcal{A}}_*$ detects equivalences, it follows that $\sigma^0_H \simeq X$ as claimed. $\square$

12.A. **Resolutions of universal spaces.** Although the vanishing result we require is purely algebraic, it is illuminating to give a geometric realization. In later subsections we will only use the algebraic exact sequences that arise by applying $\pi^{\mathcal{A}}_*$ to the sequences of spectra.

Proposition 12.2. *If* $\dim(G/H) = d$ *there is a sequence*

$$G/H_+ \longrightarrow EG/H_+ \longrightarrow \binom{d}{1}\Sigma^2 EG/H_+ \longrightarrow \binom{d}{2}\Sigma^4 EG/H_+ \longrightarrow \cdots \longrightarrow \Sigma^{2d} EG/H_+$$

inducing an exact sequence in $\pi^{\mathcal{A}}_*$.

Proof: Note the individual entries $\pi^{\mathcal{A}}_*(X)(K)$ are obtained by first passing to geometric K-fixed points, and then extending scalars. Next observe that for both G/H_+ and EG/H_+ these fixed points are either both contractible or both copies of the corresponding spaces for the quotient group G/K. It therefore suffices to deal with the special case $H = 1$ and show that it is exact in homotopy.

The proof is based on algebra over the polynomial ring of $H^*(BG)$. Indeed, we may form an injective resolution of $\mathbb{Q}$ as the Matlis dual of the Koszul resolution for the polynomial ring of $H^*(BG)$.

First note that $[EG_+, EG_+]^G_* = H^*(BG)$ is a polynomial ring on r generators $x_1, \ldots, x_r$ of degree 2. The augmented Koszul complex takes the form

$$\mathbb{Q} \longleftarrow F_0 \longleftarrow F_1 \longleftarrow F_2 \longleftarrow \cdots \longleftarrow F_r,$$

where

$$F_i = \binom{r}{i} \Sigma^{2i} H^*(BG),$$

and is exact. Dualizing, with respect to $\mathbb{Q}$ we obtain

$$\mathbb{Q} \longrightarrow F_0^\vee \longrightarrow F_1^\vee \longrightarrow F_2^\vee \longrightarrow \cdots \longrightarrow F_r^\vee.$$

Now using the identification of $[EG_+, EG_+]^G_*$ we realize this as

$$\sigma_H^0 \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \cdots \longrightarrow I_r$$

where

$$I_i = \binom{r}{i} \Sigma^{2i} EG_+.$$

In this case (i.e., with $H = 1$), the exactness of the dual Koszul complex immediately gives exactness in the category $\mathcal{A}$. □

Now we need to deal with universal spaces $E\mathcal{F}_+$, and will construct a resolution using a filtration of $\mathcal{F}$. To describe one step, let fmax($\mathcal{F}$) consist of the subgroups of finite index in a maximal subgroup of $\mathcal{F}$, and take $\mathcal{F}^1 := \mathcal{F} \setminus \mathrm{fmax}(\mathcal{F})$. Finally, we define the codimension filtration of $\mathcal{F}$ by $\mathcal{F}^i = (\mathcal{F}^{i-1})^1$.

Proposition 12.3. *The Adams resolution of $E\mathcal{F}_+$ is*

$$E\mathcal{F}_+ \longrightarrow \bigvee_{H \in \mathrm{fmax}(\mathcal{F})} E\langle H \rangle \longrightarrow \bigvee_{H \in \mathrm{fmax}(\mathcal{F}^1)} \Sigma^1 E\langle H \rangle \longrightarrow \cdots \longrightarrow \bigvee_{H \in \mathrm{fmax}(\mathcal{F}^r)} \Sigma^r E\langle H \rangle$$

Proof: It is enough to deal with a single step since the general case follows by repeating it.

Lemma 12.4. *There is a cofibre sequence*

$$E\mathcal{F}_+ \longrightarrow \bigvee_{H \in \mathrm{fmax}(\mathcal{F})} E\langle H \rangle \longrightarrow \Sigma E\mathcal{F}^1_+,$$

which induces a short exact sequence in $\pi^{\mathcal{A}}_*$.

Proof: To see the cofibre sequence exists we need only show that the cofibre C of the universal map $E\mathcal{F}^1_+ \longrightarrow E\mathcal{F}_+$ splits as a wedge as indicated. Indeed, we may construct a map $C \longrightarrow E\langle H \rangle$ which is an equivalence on H-fixed points for each $H \in \mathrm{fmax}(\mathcal{F})$ by obstruction theory, and hence a map $C \longrightarrow \bigvee_{H \in \mathrm{fmax}(\mathcal{F})} E\langle H \rangle$, since the sum coincides with the product up to homotopy. It is an equivalence by the Whitehead Theorem. To see that the cofibre sequence gives an algebraic short exact sequence we observe that $E\mathcal{F}^1_+ \longrightarrow E\mathcal{F}_+$ must be zero, since $\pi^{\mathcal{A}}_*(E\mathcal{F}^1_+)$ is fmax($\mathcal{F}$)-torsion in the sense that if $H \in \mathcal{F}^1$ lies in $K \in \mathrm{fmax}(\mathcal{F})$ then

$$\mathcal{E}^{-1}_{H/K} \pi^{\mathcal{A}}_*(E\mathcal{F}^1_+)(H) = \pi^{\mathcal{A}}_*(E\mathcal{F}^1_+)(K) = 0,$$

whereas $\mathcal{E}_{H/K}$ consists of non-zero divisors in $\pi^{\mathcal{A}}_*(E\mathcal{F}_+)(H)$. □

□

12.B. **Algebraic connectivity.** It is convenient to have some terminology to discuss the algebraic counterparts of natural homotopy theoretic constructions.

Definition 12.5. (i) We say that a spectrum X is *algebraically c-connected at H* if

$$\mathrm{Ext}_{\mathcal{A}}^{s,t}(\pi_*^{\mathcal{A}}(\sigma_H^0), \pi_*^{\mathcal{A}}(X)) = 0 \text{ for } t-s \leq c.$$

If X is c-connected but not $(c+1)$-connected, we write $\mathrm{algconn}_H(X) = c$, and if it is c-connected for all integers c we write $\mathrm{algconn}_H(X) = \infty$.

(ii) We say that X has *vanishing line of slope 1* at H if $\mathrm{Ext}_{\mathcal{A}}^{s,t}(\pi_*^{\mathcal{A}}(\sigma_H^0), \pi_*^{\mathcal{A}}(X)) = 0$ for $t-s \leq s + \mathrm{algconn}_H(X)$.

Remark 12.6. (i) If X has algebraic H-connectivity c then $\Sigma^n X$ has algebraic H-connectivity $c+n$.

(ii) If X has vanishing line of slope 1 then in particular the non-vanishing Ext group is on the 0-line.

(iii) By the Adams spectral sequence, the first non-zero σ_H^0-homotopy group of X is in dimension $\mathrm{algconn}_H(X)+1$, and if X has vanishing line of slope 1 then

$$[\sigma_H^0, X]^G_{\mathrm{algconn}_H(X)+1} = \mathrm{Hom}_{\mathcal{A}}(\pi_*^{\mathcal{A}}(\sigma_H^0), \pi_*^{\mathcal{A}}(X)).$$

We may manipulate algebraic connectivity with a slope 1 vanishing line, rather like homotopy groups.

Lemma 12.7. *If the sequence of spectra $X \longrightarrow Y \longrightarrow Z$ induces a short exact sequence*

$$0 \longrightarrow \pi_*^{\mathcal{A}}(X) \longrightarrow \pi_*^{\mathcal{A}}(Y) \longrightarrow \pi_*^{\mathcal{A}}(Z) \longrightarrow 0$$

where Y and Z have vanishing lines of slope 1 and $\mathrm{algconn}_H(Y) = c, \mathrm{algconn}_H(Z) = c+2$ *then X has a vanishing line of slope 1 and* $\mathrm{algconn}_H(X) = c$. □

This allows us to give estimates using resolutions.

Corollary 12.8. *If the sequence of spectra*

$$X \longrightarrow Y_0 \longrightarrow Y_1 \longrightarrow Y_2 \longrightarrow \cdots$$

induces an exact sequence

$$0 \longrightarrow \pi_*^{\mathcal{A}}(X) \longrightarrow \pi_*^{\mathcal{A}}(Y_0) \longrightarrow \pi_*^{\mathcal{A}}(Y_1) \longrightarrow \pi_*^{\mathcal{A}}(Y_2) \longrightarrow \cdots$$

where Y_i has a vanishing line of slope 1 and $\mathrm{algconn}_H(Y_i) = c+2i$ *then X has a vanishing line of slope 1 and* $\mathrm{algconn}_H(X) = c$. □

12.C. **Estimates of connectivity.** Finally we may use the algebraic resolutions from Subsection 12.A to estimate algebraic connectivity. The starting point is the fact that $E\langle K\rangle$ is injective so it is easy to determine connectivity, for example by using the known homotopy groups of universal spaces.

Lemma 12.9. *The spectrum $E\langle K\rangle$ has a slope 1 vanishing line and*

$$\mathrm{algconn}_H(E\langle K\rangle) + 1 = \begin{cases} \dim(H/K) & \text{if } K \subseteq H \\ \infty & \text{if } K \not\subseteq H. \end{cases} \quad \square$$

From this we may prove the required result by connecting σ^0_H to objects $E\langle K\rangle$ by the cofibre sequences of Subsection 12.A, and applying 12.8. One expects to be able to apply 12.8 only when the bottom homotopy comes from the displayed sequences, and not from the connecting homomorphism.

Lemma 12.10. *The spectrum EG/K_+ has a slope 1 vanishing line and*

$$\mathrm{algconn}_H(EG/K_+) + 1 = \dim(H/K) \text{ if } K \subseteq H.$$

Proof: We use the resolution described in 12.3. The family in question consists of subgroups of K. Accordingly the ith term in the codimension filtration consists of groups of codimension i in K. Thus the algebraic connectivity of a term $\Sigma^i E\langle L\rangle$ in the ith stage of the resolution is $\dim(H/L)+i = \dim(H/K)+2i$ if $L \subseteq H$ (and ∞ otherwise). This provides the hypothesis we need to apply 12.8. □

The Koszul-type resolution of 12.2 gives the required estimate for G/K_+.

Corollary 12.11. *The spectrum G/K_+ has a slope 1 vanishing line and*

$$\mathrm{algconn}_H(G/K_+) + 1 = \dim(H/K) \text{ if } K \subseteq H. \quad \square$$

If H is connected then $\sigma^0_H = G/H_+$, and in general σ^0_H is a retract of G/H_+, so we obtain the estimate required for Theorem 12.1.

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