MAS435: ALGEBRAIC TOPOLOGY
2016-17
EXERCISE SHEET 3

1. Consider self-homeomorphisms f of a sphere S^n. Show that for each n there is such a map f with no fixed points. What is $f_* : H_n(S^n) \longrightarrow H_n(S^n)$?

Deduce that if n is even, there is no free action of the cyclic group of order 4 on S^n. What if n is odd?

2. Consider self-maps of a compact connected orientable surface $M(g)$ of genus g.

(i) What is the Euler characteristic of $M(g)$? For which g is there a self-map homotopic to the identity without fixed points?

(ii) Suppose that $f : M(g) \longrightarrow M(g)$ has the property that $f_* : H_1(M(g)) \longrightarrow H_1(M(g))$ is multiplication by -1. You may assume without proof that this implies that f_* is the identity on $H_0(M(g))$ and $H_2(M(g))$. What is the Lefschetz number of f? Can you think of a continuous involution f (i.e., a map f with $f^2 = id$) with this number of fixed points?

(iii) Show that if g is odd, there is a continuous involution f of $M(g)$ with no fixed points. What is the trace of $f_* : H_1(M(g)) \longrightarrow H_1(M(g))$.

(iv) Show that if $g = 3n + 1$ there is a self-homeomorphism f of $M(g)$ with $f^3 = id$ and no fixed points. What is the trace of $f_* : H_1(M(g)) \longrightarrow H_1(M(g))$.

3. Consider self-homeomorphisms f of the complex projective plane $\mathbb{C}P^2$. Show that any such f has a fixed point.

Show that the same argument applies to $\mathbb{C}P^{2n}$. Show that there is a self-map of $\mathbb{C}P^1$ without a fixed point.

4. Consider self-homeomorphisms f of the real projective plane $\mathbb{R}P^2$. Show that any such f has a fixed point.

Show that there are self maps of $\mathbb{R}P^1$ and $\mathbb{R}P^3$ without fixed points. Can these maps be chosen to be homotopic to the identity?

5. Suppose that X is a connected n-manifold, and $f : X \longrightarrow X$ has a single fixed point. Assuming that X can be triangulated as a finite simplicial complex, show that the Lefschetz number of f is equal to 1. [Assume any properties of the triangulation that are convenient.]

School of Mathematics and Statistics, Hicks Building, Sheffield S3 7RH. UK.
E-mail address: j.greenlees@sheffield.ac.uk