MAS435: ALGEBRAIC TOPOLOGY
2016-17
EXERCISE SHEET 2

1. Suppose X is a space and τ is a path from x_1 to x_2. Note that if ω is a loop based at x_1 then

$$\hat{t}(\omega) := \overline{\tau} \cdot \omega \cdot \tau$$

is a loop based at x_2.

(i) Show that if $\omega \simeq \omega'$ then $\hat{t}(\omega) \simeq \hat{t}(\omega')$ (both *path* homotopies). Deduce that $\hat{t}_\tau$ induces a map

$$t_\tau : \pi_1(X, x_1) \longrightarrow \pi_1(X, x_2).$$

(ii) Show that $\hat{t}_\tau$ is a group homomorphism.

(iii) Show that if σ is a path from x_0 to x_1

$$\hat{t}_{\sigma\cdot\tau}(\omega) \simeq \hat{t}_\tau(\hat{t}_\sigma(\omega))$$

(iv) Show that $\hat{t}_{c_{x_0}}(\omega) \simeq \omega$ and hence that $t_{c_{x_0}} = id$.

(v) Show that if $\tau \simeq \tau'$ then $\hat{t}_\tau(\omega) \simeq \hat{t}_{\tau'}(\omega)$, so that $t_\tau = t_{\tau'}$

(vi) Deduce from Parts (i)-(v) that t_τ and $t_{\overline{\tau}}$ are inverse isomorphisms, and in particular

$$\pi_1(X, x_0) \cong \pi_1(X, x_1).$$

(vii) Give an example where $\pi_1(X, x_0) \not\cong \pi_1(X, x_1)$.

2. Suppose G is a topological group (i.e., G is a topological space and a group, and the multiplication map $\mu : G \times G \longrightarrow G$ and the inverse map $i : G \longrightarrow G$ are continuous).

(i) Show that S^1 is a topological group.

(ii) Show that the set $O(n) = \{A \in \mathcal{M}_n(\mathbb{R}) \mid AA^t = I\}$ of $n \times n$ orthogonal matrices is a topological group.

(iii) Define a new operation $*$ on loops based at the identity $e \in G$ by

$$(\omega * \sigma)(t) := \mu(\omega(t), \sigma(t)).$$

Observe that $\omega * \sigma$ is continuous and hence a loop based at e.

(iv) Show that there are path homotopies

$$\sigma \cdot \omega \simeq \sigma * \omega \simeq \omega \cdot \sigma.$$

Conclude that $\pi_1(G, e)$ is abelian.

(v) Does the Klein bottle admit the structure of a topological group?

3. Show that if G is a topological group then $\pi_1(G, g) \cong \pi_1(G, e)$ for all $g \in G$. [Hint: Don't try to use Question 1.]

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