7 The fundamental group of the circle and covering spaces

7.1 An informal sketch for the circle

The fundamental group of the circle is $\mathbb{Z}$, but we haven't proved it yet. This result is intuitively clear, but the proof needs some new ideas. Here are some remarkable corollaries:

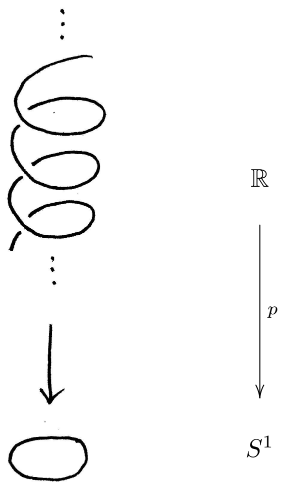
The fundamental theorem of algebra: *Every non-constant polynomial with coefficients in $\mathbb{C}$ has a root in $\mathbb{C}$.*
The Brouwer fixed point theorem: *Every continuous self-map of the closed unit disc has a fixed point.*
The Borsuk-Ulam theorem in dimension 2: *For every continuous map $f : S^2 \longrightarrow \mathbb{R}^2$ there is a pair of antipodal points $x, -x \in S^2$ such that $f(x) = f(-x)$.*
Corollary:*If S^2 is expressed as the union of three closed sets then at least one must contain a pair of antipodal points.*

The key idea is as follows:

When you're walking up the stairs in the Hicks Building, it is easier to look at the signs to see what floor you've reached than to count how many times you've gone round!

Sketch proof that $\pi_1(S^1) = \mathbb{Z}$

We get a bit "confused" about going round our circle multiple times, so let's "unwind" our circle:



Now, instead of counting how many times we went round, we count count how many "floors" up (or down) we've gone. We can choose the map p so that the preimages of the basepoint x are precisely the integers. So the number of times we've gone round the circle is precisely counted by what number we land on in the spiral version. In other words, I define a map $\ell : \pi_1(S^1, 1) \longrightarrow \mathbb{Z}$ as follows. If my original loop is ω, then as the path moves along ω, I copy it on the stairs. When I have finished copying ω I am standing above the basepoint again. This means I am at some integer $n \in \mathbb{Z} \subset \mathbb{R}$, and

$$\ell([\omega]) = n.$$

There is a bit of work to be done to check this always works, that it is well defined, and that it gives a group homomorphism. We will develop this machinery in the rest of this chapter.

7.2 Covering spaces

We constantly consider the example $e : \mathbb{R} \longrightarrow S^1$ defined by $p(t) = e^{2\pi i t}$. The crucial features are embodied in the following two lemmas. First, $\mathbb{R}$ has a special property that means it can be used to calculate all of the fundamental group.

Lemma 7.1. *Any two paths from a to b in $\mathbb{R}$ are path homotopic.*

Proof. We have seen this is immediate by using a linear homotopy. □

More directly relevant is the property enjoyed by the map p.

Lemma 7.2. *(a)* $e(t) = 1 \iff t \in \mathbb{Z}$.

(b) For any $z \in S^1$ *there is a neighbourhood* U *of* z *so that*

$$e^{-1}(U) \cong U \times \mathbb{Z}$$

and

$$e|_{\tilde{U}_n} : \tilde{U}_n \xrightarrow{\cong} U.$$

Proof. (a) Clear.

(b) We could jsut argue that there is a locally defined *log* function. Probably simpler to say $e^{-1}(e^{2\pi it}) = \mathbb{Z} + t$ so that

$$e^{-1}(S^1 \setminus \{e^{2\pi it}\}) = \mathbb{R} \setminus (\mathbb{Z} + t) = \coprod_n (n + t, n + t + 1).$$

□

These properties have names.

Definition 7.3. *Consider a continuous function* $p : \tilde{X} \longrightarrow X$.

(i) We say that an open set $U \subseteq X$ *is* evenly covered *if*

$$p^{-1}(U) = \coprod_{\alpha \in A} \tilde{U}_\alpha$$

for open subsets $\tilde{U}_\alpha \subseteq \tilde{X}$ *and*

$$p|_{\tilde{U}_\alpha} : \tilde{U}_\alpha \xrightarrow{\cong} U$$

for all $\alpha \in A$.

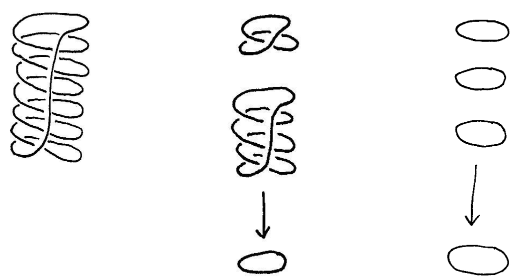
(ii) We that p *is a* covering map *(and* $\tilde{X}$ *is a* covering space *of* X*) if every point* $x \in X$ *has an evenly covered neighbourhood.*

7.3 Some pictures of coverings

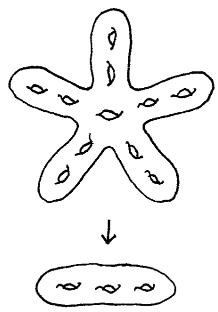
Recall that we tried to calculate the fundamental group of S^1 by thinking about a helix mapping onto the circle:

This is a good prototype example of what we will call a covering space. We are "unwinding" the loops in the space.

Here are some more covering spaces of S^1:



Here some other covering spaces. The first two cover the space  ?



We will attempt to find *all* the covering spaces of a space. The main answer is a correspondence

connected covering spaces	$\longleftrightarrow$	subgroups of the fundamental group

This might remind you of Galois theory, in which you have a correspondence between

field extensions	$\longleftrightarrow$	subgroups of the Galois group

A covering space sort of "unloops" some of the loops in your space. The corresponding subgroup of the fundamental group consists of the loops that you *haven't* unlooped yet.

As your covering space gets bigger and bigger, your corresponding subgroup gets smaller and smaller, and finally you get the **universal cover** corresponding to the **trivial subgroup**. This is where everything has been maximally unwound—you no longer have any loops left.

7.4 The fundamental group of the circle

We are now ready to give the calculation of the fundamental group of the circle. First of all, let us define some loops:

$$\omega_n : [0,1] \longrightarrow S^1$$

is defined by

$$\omega_n(s) = e^{2\pi i n s}.$$

We note that ω_n is a composite of the linear path from 0 to n in $\mathbb{R}$ and the exponential map:

$$\begin{array}{ccc} [0,1] & \xrightarrow{\tilde{\omega}_n} & \mathbb{R} \\ & \searrow^{\omega_n} & \downarrow e \\ & & S^1 \end{array} .$$

Theorem 7.4. *The map*

$$\theta : \mathbb{Z} \xrightarrow{\cong} \pi_1(S^1, 1)$$

defined by

$$\theta(n) = [\omega_n]$$

is a group isomorphism.

Proof. We are going to give the proof in three stages.

Step 1: The first stage is to note that θ is a homomorphism. Indeed, we have seen that any two paths between the same points of $\mathbb{R}$ are homotopic. Hence if we write l_x^y for the linear path from x to y (i.e., $l_x^y(s) = (1-s)x + sy$) we have a path homotopy

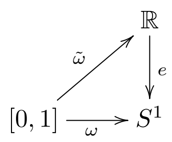
$$l_x^y \cdot l_y^z \simeq l_x^z.$$

Now note $\omega_n = e \circ l_p^{p+n}$ for any integer p and

$$\omega_m \cdot \omega_n = e \circ (l_0^n \cdot l_n^{m+n}) \simeq e \circ l_0^{m+n} = \omega_{m+n}.$$

Step 2: The map θ is onto. Indeed, if we have an element g of $\pi_1(S^1, 1)$ we want to show that there is an n so that $\theta(n) = g$. Indeed, if $g = [\omega]$ we need to show that $\omega \simeq \omega_n$. The way we do this is by counting floors in the Hicks Building.

Lemma 7.5. *(Path Lifting Lemma) Given any loop $\omega : [0,1] \longrightarrow S^1$ based at $1 \in S^1$, and an integer n there is a unique path $\tilde{\omega} : [0,1] \longrightarrow \mathbb{R}$ starting at n so* *that* $\omega = \pi \circ \tilde{\omega}$*:*



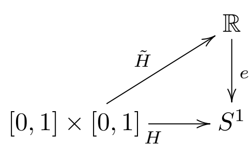
Notice that this lets us show θ is surjective. Indeed, an element $[\omega] \in \pi_1(S^1, 1)$ can be hit as follows. We lift ω from 0 to obtain $\tilde{\omega}$. This is a path from 0 to some other point n, and since $e(n) = 1$, we see n is an integer. Now $\tilde{\omega} \simeq \tilde{\omega}_n$. And

$$\omega = e \circ \tilde{\omega} \simeq e \circ \tilde{\omega}_n = \omega_n.$$

Hence $[\omega] = \theta(n)$ as required.

Step 3: The map θ is injective. This follows the usual pattern that injectivity is like the proof of surjectivity, but applied to homotopies.

Lemma 7.6. *(Homotopy Lifting Lemma) Given any homotopy* $H : [0,1] \times [0,1] \longrightarrow S^1$ *between loops* ω *and* ω' *based at* $1 \in S^1$*, and a lift* $\tilde{\omega}$ *of* ω *from* n *there is a unique homotopy* $\tilde{H} : [0,1] \times [0,1] \longrightarrow \mathbb{R}$ *of paths starting at* $\tilde{\omega}$ *so that* $H = \pi \circ H$*:*



Notice that this lets us show θ is injective. Indeed, if we suppose $\theta(m) = \theta(n)$ this means that there is a homotopy $H : \omega_m \simeq \omega_n$. The homotopy lifting lemma gives a homotopy $\tilde{H}$ from $\tilde{\omega}_m$ to another path from 0. The second path lies over ω_n and hence by uniqueness is $\tilde{\omega}_n$. Furthermore the far end of $\tilde{H}$ (namely $\tilde{H}(1, s)$) lies over 1 (since H is a path homotopy). It is therefore a path in $\mathbb{Z}$ and hence constant. Thus

$$\theta(m) = \tilde{\omega}_m(1) = \tilde{\omega}_n(1) = n.$$

□

7.5 The Brouwer fixed point theorem

After this calculation, we are due a proper application. The basis of the result is a formality.

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Lemma 7.7. *The circle S^1 is not a retract of $\overline{B}^2$.*

Proof. If A is a retract of X then $\pi_1(A, a_0)$ is a retract of $\pi_1(X, a_0)$. Thus if $\pi_1(X, a_0) = 1$ then $\pi_1(A, a_0) = 1$. In our case $\overline{B}^2 \simeq *$ has trivial fundamental group, whereas $\pi_1(S^1, 1) \cong \mathbb{Z}$. □

Theorem 7.8. (Brouwer Fixed Point Theorem) *Any map $f : \overline{B}^2 \longrightarrow \overline{B}^2$ has a fixed point.*

Proof. If $f : \overline{B}^2 \longrightarrow \overline{B}^2$ is a self-map with no fixed points then we can define a map $r : \overline{B}^2 \longrightarrow S^1$ by taking $r(P)$ to be the place where the line from $f(P)$ to P meets the circle. If P is on the circle then $r(P) = P$, so it is a retraction.

One may argue that r is continuous by starting from the fact that if f has no fixed point, then by compactness, there is an $m > 0$ so that $||P - f(P)|| \geq m$. After this, a little trigonometry permits an ϵ-δ proof. □

7.6 The Lebesgue Covering Lemma

The following result really belongs under the discussion of compactness, but it seemed worth saving it until we needed it.

Definition 7.9. *If X is a metric space and $\mathfrak{U}$ is a cover of X, we say that δ is a* Lebesgue number *for $\mathfrak{U}$ if for any $x \in X$ the closed ball $\mathring{B}_\delta(x)$ lies in one of the sets of $\mathfrak{U}$.*

Remark 7.10. If δ is a Lebesgue number of $\mathfrak{U}$ and $\delta' \leq \delta$ then δ' is also a Lebesgue number for $\mathfrak{U}$.

Lemma 7.11. *If K is a compact metric space and $\mathfrak{U}$ is an open cover of K, there is Lebesgue number $\delta > 0$ for $\mathfrak{U}$.*

Proof. For each $x \in X$, there is a set $U(x) \in \mathfrak{U}$ containing x and since $U(x)$ is open, there is a number ϵ_x with $\mathring{B}_\epsilon(x) \subseteq U(x)$. Consider the new open cover $\{\mathring{B}_{\epsilon_x/2}(x) \mid x \in K\}$ of K. Since K is compact, it has a finite subcover

$$\{\mathring{B}_{\epsilon_{x_1}/2}(x_1), \mathring{B}_{\epsilon_{x_2}/2}(x_2), \ldots, \mathring{B}_{\epsilon_{x_N}/2}(x_N)\},$$

and we take

$$\lambda := \min\{\epsilon_{x_1}/2, \ldots, \epsilon_{x_N}/2\}$$

We claim this is a Lebesgue number for $\mathcal{U}$. Indeed if $x \in X$, we have $x \in \mathring{B}_{\epsilon_{x_i}/2}$ (x_i) for some i and then

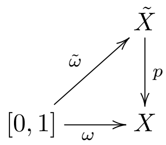
$$\mathring{B}_{\epsilon_x/2}\,(x) \subseteq \mathring{B}_{\epsilon_{x_i}}\,(x_i) \subseteq U(x_i)$$

as required. □

7.7 Path lifting lemma

Let us restate the path lifting lemma for a general covering map $p : \tilde{X} \longrightarrow X$ and prove it. You may wish to think of just the example $\mathbb{R} \longrightarrow S^1$.

Lemma 7.12. *(Path Lifting Lemma) [**Given:** ω **and** $\tilde{x}_0$**. Get: a unique** $\tilde{\omega}$] Suppose given any loop $\omega : [0,1] \longrightarrow X$ based at $x_0 \in X$, and a point $\tilde{x}_\alpha$ over x_0. There is a unique path $\tilde{\omega} : [0,1] \longrightarrow \tilde{X}$ starting at $\tilde{x}_\alpha$ so that $\omega = \pi \circ \tilde{\omega}$:*



Proof. Choose a cover $\mathcal{U}$ of X by evenly covered neighbourhoods This gives an open cover $\omega^{-1}\mathcal{U}$ of $[0,1]$. By the Lebesgue covering lemma, there is an n so that any interval of length $1/n$ lies entirely in one of the sets.

Now divide $[0,1]$ into n intervals

$$[0/n, 1/n], [1/n, 2/n], [2/n, 3/n], \ldots, [(n-1)/n, n/n].$$

We will define $\tilde{\omega}$ on each of these in succession. Indeed, if $\tilde{\omega}$ is defined on $[0/n, k/n]$ then we may extend it over $[k/n, (k+1)/n]$ as follows. It is already defined on k/n by hypothesis. Now consider the evenly covered neighbourhood containing $\omega(k/n)$. Then choose $\tilde{U}_\alpha$ containing $\tilde{\omega}(k/n)$ and define

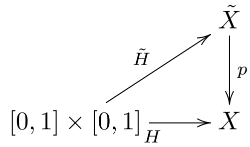
$$\omega(t/n + s) = (p|_{\tilde{U}_\alpha})^{-1}(\omega(t/n + s))$$

Repeating this n times, we have $\tilde{\omega}$ defined on all of $[0,1]$. □

7.8 The homotopy lifting lemma

Let us restate the homotopy lifting lemma for a general covering map $p : \tilde{X} \longrightarrow X$ and prove it. You may wish to think of just the example $\mathbb{R} \longrightarrow S^1$.

Lemma 7.13. *(Homotopy Lifting Lemma)* ***[Given:*** $H, \tilde{\omega}, \tilde{\omega}'$***. Get:*** $\tilde{H}$ ***and*** $\tilde{\omega}(1) = \tilde{\omega}'(1)$***]*** *Given any homotopy* $H : [0,1] \times [0,1] \longrightarrow S^1$ *between loops* ω *and* ω' *based at* $x_0 \in X$*, and lifts* $\tilde{\omega}$ *and* $\tilde{\omega}'$ *from* $\tilde{x}_0$ *there is a unique homotopy* $\tilde{H} : [0,1] \times [0,1] \longrightarrow \tilde{X}$ *of paths starting at* $\tilde{\omega}$ *so that* $H = \pi \circ H$*:*



Proof. Choose a cover $\mathcal{U}$ of X by evenly covered neighbourhoods This gives an open cover $\omega^{-1}\mathcal{U}$ of $[0,1]$. By the Lebesgue covering lemma, there is an n so that any square of side $1/n$ lies entirely in one of the sets. Now divide $[0,1] \times [0,1]$ into n^2 squares

$$[i/n, (i+1)/n] \times [j/n, (j+1)/n].$$

We will define $\tilde{H}$ on each of these in succession, starting with $i = j = 0$, and working along the jth row of squares with i increasing then moving to the next j row and so forth. When we reach a new square, $[i/n, (i+1)/n] \times [j/n, (j+1)/n]$, the homotopy H maps the square into an evenly covered neighbourhood U. There are two cases.

Case 1: Until the top row, we have $\tilde{H}$ defined already on the bottom and the left hand edge.

Case 2: In the top row, we have $\tilde{H}$ defined already on the left hand edge, the top and the bottom.

In either case, the part where $\tilde{H}$ is already defined is path connected, and hence $\tilde{H}$ maps it into a single $\tilde{U}_\alpha$. Accordingly, we may define $\tilde{H}$ on the entire square by the formula

$$\tilde{H} = (p|_{\tilde{U}_\alpha})^{-1} \circ H.$$

Repeating this n^2 times we have $\tilde{H}$ defined on all of $[0,1] \times [0,1]$. ∎

8 The Seifert-van Kampen theorem

We have seen that to calculate the fundamental group of the circle it was very useful to know that $\mathbb{R}$ had trivial fundamental group. Rather embarassingly, the only method we have so far for proving a space has trivial fundamental group is to show the space is contractible.

Remark 8.1. A space X is called *simply connected* if $\pi_1(X, x_0) = 1$ for all bassepoints x_0. A space which is path connected and simply connected is called *1-connected.* [Warning: some authors use 'simply connected' to imply a space is path connected as well.]

8.1 Decomposing loops

Suppose then that we have a space X written as a union of two open subspaces

$$X = A \cup B.$$

We would like to understand the fundamental group of X in terms of the fundamental groups of A and B. We can do this rather generally, but there are some hypotheses.

Theorem 8.2. *Suppose $X = A \cup B$, with A and B open, and suppose $x_0 \in A \cap B$. If $A \cap B$ is path connected then $\pi_1(X, x_0)$ is generated by loops entirely in A and loops entirely in B.*

Corollary 8.3. *With the hypotheses of the theorem and if $\pi_1(A, x_0)$ and $\pi_1(B, x_0)$ are both trivial then $\pi_1(X, x_0)$ is also trivial.* □

Example 8.4. *$\pi_1(S^n)$ is trivial for any $n \geq 2$. Indeed, we can take $A = S^n \setminus \{N\}, B = S^n \setminus \{S\}$. Then $A \simeq *, B \simeq *$ and $A \cap B \cong S^{n-1} \times (-1, 1)$.*

Proof. We will show that any loop $\omega : [0, 1] \longrightarrow X$ is homotopic to a concatenation of loops mapping entirely into A or entirely into B.

Consider the open cover $\{A, B\}$ of X and a loop $\omega : [0, 1] \longrightarrow X$. Then $\{\omega^{-1}(A), \omega^{-1}(B)\}$ is a cover of the compact metric space $[0, 1]$ by the Lebesgue Covering Lemma, there is a number n so that any interval of length n lies in one of the two sets.

Now, breaking ω up into n shorter paths

$$\omega = \omega_0 \cdot \omega_1 \cdot \cdots \cdot \omega_{n-1}$$

where ω_i is ω restricted to $[i/n, (i+1)/n]$. Since $1/n$ is less than the Lebesgue number, we find each ω_i maps entirely into A or into B. Now we group together the ω_i mapping into A and B. Indeed, if we suppose (withoug loss of generality) that ω_0 maps into A, we take

$$\hat{\omega}_0 = \omega_0 \cdot \omega_1 \cdot \cdots \cdot \omega_i$$

where ω_{i+1} is the first path that does not map into A, and then proceed with $\hat{\omega}_2$ mapping into B and so forth, until we reach

$$\omega \simeq \hat{\omega}_0 \cdot \cdots \cdot \hat{\omega}_k$$

where the $\hat{\omega}_i$ map alternately entirely into A or entirely into B.

Now we name the endpoints, so that $\hat{\omega}_i$ goes from y_i to y_{i+1}. Since y_i is an endpoint of both $\hat{\omega}_{i-1}$ and $\hat{\omega}_i$, we find $y_i \in A \cap B$. We also have $y_0 = y_{k+1} = x_0$. Since $A \cap B$ is path connected, we can choose paths τ_i from x_0 to y_i in $A \cap B$. Finally

$$\omega \simeq \hat{\omega}_0 \cdot \cdots \cdot \hat{\omega}_k \simeq \tau_0 \cdot \hat{\omega}_0 \cdot \overline{\tau}_1 \cdot \tau_1 \cdot \cdots \cdot \overline{\tau}_k \cdot \tau_k \cdot \hat{\omega}_k \cdot \overline{\tau}_{k+1} \simeq \gamma_0 \cdot \gamma_1 \cdot \cdots \cdot \gamma_k$$

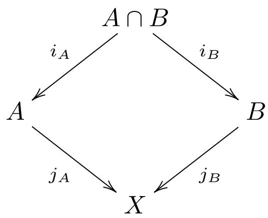
where

$$\gamma_i = \tau_i \cdot \hat{\omega}_i \cdot \overline{\tau}_{i+1}$$

is a loop entirely in A or entirely in B. □

8.2 Generators and relations

We have shown that $\pi_1(X, x_0)$ is generated by $\pi_1(A, x_0)$ and $\pi_1(B, x_0)$. It is also clear that if we have a loop ω in $A \cap B$ then it means the same as a loop in X whether we view it as having come from A or from B. More precisely, if the maps are as follows



then it is obvious that $j_A \circ i_A = j_B \circ i_B$.

It turns out that this is the only requirement.

Theorem 8.5. *Suppose $X = A \cup B$ with A and B open, and $x_0 \in A \cap B$. If $A \cap B$ is path connected and the fundamental groups of A, B and $A \cap B$ are given by generators and relations as follows*

$$\pi_1(A, x_0) = \langle \alpha_1, \ldots, \alpha_m \mid \rho_1, \ldots, \rho_p \rangle, \pi_1(B, x_0) = \langle \beta_1, \ldots, \beta_n \mid \sigma_1, \ldots, \sigma_q \rangle$$

and

$$\pi_1(A \cap B, x_0) = \langle \gamma_1, \ldots, \gamma_l \mid \textit{unnamed} \rangle$$

then the fundamental group of X is given by the formula

$$\pi_1(X, x_0) = \langle \alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_n \mid \rho_1, \dots, \rho_p, \sigma_1, \dots, \sigma_q, (i_A)_*\gamma_1 = (i_B)_*\gamma_1, \dots, (i_A)_*\gamma_l = (i_B)_*\gamma_l \rangle$$

Symbolically, one writes

$$\pi_1(X, x_0) = \pi_1(A, x_0) *_{\pi_1(A\cap B, x_0)} \pi_1(B, x_0).$$

We will not give the proof here, but we note that we have already established the statement about generators and that all the given relations hold. It remains only to check these generate all relations. In the usual way, this is given by using the same argument again, but applied to homotopies rather than loops.

Example 8.6. *The fundamental group of a bouquet of n circles is the free group on n generators:*

$$\pi_1(\bigvee_1^n S^1) = \langle x_1, \dots, x_n \rangle.$$

Example 8.7. *The fundamental group of the compact connected orientable surface $M(g)$ of genus g is*

$$\pi_1(M(g)) = \langle x_1, \dots, x_g, y_1, \dots, y_g \mid [x_1, y_1] \cdots [x_g, y_g] \rangle,$$

where

$$[x, y] = x^{-1}y^{-1}xy.$$

9 Group actions and the fundamental group

The point of this section is to show the power of the method we used to calculate the fundamental group of the circle. It applies much more generally with no extra effort.

9.1 The floor label map

The key idea of that was that to understand a loop we walk above the loop on the stairs and then record the name of the floor you reach at the end.

More generally, we suppose $p : \tilde{X} \longrightarrow X$ is a covering map, $x_0 \in X$ is a basepoint and $\tilde{x}_0 \in \tilde{X}$ lies over x_0, and write $F = p^{-1}(x_0)$ for the set of points over x_0 (the set of 'floor labels')

Definition 9.1. *The floor label map*

$$\ell : \pi_1(X, x_0) \xrightarrow{\cong} F$$

is defined by

$$\ell([\omega]) := \tilde{\omega}(1).$$

Here ω is a loop in X based at x_0, and $\tilde{\omega}$ is the unique lift of ω to a path starting from $\tilde{x}_0$, as guaranteed by the Path Lifting Lemma.

Remark 9.2. We note that the map ℓ obviously depends on the starting point $\tilde{x}_0$, and we may write $\ell_{\tilde{x}_0}$ if we want to emphasize this.

Theorem 9.3. *If $\tilde{X}$ is path connected and $\pi_1(\tilde{X}, \tilde{x}_0) = 1$ then the floor label map*

$$\ell : \pi_1(X, x_0) \xrightarrow{\cong} F$$

is a bijection.

Proof. ℓ is surjective: If $\tilde{x}_1 \in F$ then by path connectedness of $\tilde{X}$ we may choose a path $\tilde{\omega}$ from $\tilde{x}_0$ to $\tilde{x}_1$. Since $\tilde{x}_i \in F$ the path $\omega := p \circ \tilde{\omega}$ is a loop, and by construction $\tilde{\omega}$ is a lift of ω starting from $\tilde{x}_0$. Hence $\ell([\omega]) = \tilde{x}_1$.

ℓ is injective: As usual injectivity is essentially surjectivity on homotopies. Let us suppose that $\tilde{\omega}$ and $\tilde{\omega}'$ are the lifts of ω and ω' from $\tilde{x}_0$. Indeed, if

$$\ell([\omega]) = \tilde{\omega}(1) = \tilde{x}_1 = \tilde{\omega}'(1) = \ell([\omega']).$$

both $\tilde{\omega}$ and $\tilde{\omega}'$ are paths from $\tilde{x}_0$ to $\tilde{x}_1$. We can therefore make a loop based at $\tilde{x}_0$ by concatenating $\tilde{\omega}$ and the reverse of $\tilde{\omega}'$, though it is convenient to interpolate a constant loop. In any case, since $\pi_1(\tilde{X}, \tilde{x}_0)$, there is a nullhomotopy $\tilde{H}$ of the loop $\tilde{\omega} \cdot (c_{\tilde{x}_1} \cdot \overline{\tilde{\omega}'})$. We may depict this as follows:

Now, we can redraw the square $[0, 1] \times [0, 1]$ (i.e., use a homeomorphism) so that the three regions of the boundary are as indicated:

This gives a homotopy from $\tilde{\omega} \simeq \tilde{\omega}'$ as required. ∎

For very small groups the number of elements determines the group.

Example 9.4. *For any $n \geq 2$ choice of basepoint, $\pi_1(\mathbb{R}P^n)$ is a cyclic group of order 2.*

Indeed, we have the covering map $p : S^n \longrightarrow \mathbb{R}P^n$, and F has two elements.

Corollary 9.5. *(Borsuk-Ulam Theorem) There is no map $f : S^2 \longrightarrow S^1$ with the property $f(-P) = -f(P)$.*

Proof. If f exists, we may form the square

$$\begin{array}{ccc} S^2 & \xrightarrow{f} & S^1 \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle p_1} \\ \mathbb{R}P^2 & \xrightarrow[\bar{f}]{} & \mathbb{R}P^1 \end{array}$$

Applying π_1 we obtain

$$\begin{array}{ccc} \pi_1(S^2) & \xrightarrow{f_*} & \pi_1(S^1) \\ {\scriptstyle (p_2)_*}\downarrow & & \downarrow{\scriptstyle (p_1)_*} \\ \pi_1(\mathbb{R}P^2) & \xrightarrow[\bar{f}_*]{} & \pi_1(\mathbb{R}P^1) \end{array}$$

Along the bottow row we have

$$\mathbb{Z}/2 \longrightarrow \mathbb{Z},$$

which is the trivial map (since $\mathbb{Z}/2$ is torsion and $\mathbb{Z}$ is torsion free). On the other hand, if we choose a generating loop in $\pi_1(\mathbb{R}P^2)$ then it lifts to a path in S^2 joining two antipodal points P and $-P$. The image under f also joins two antipodal points by hypothesis, and hence maps to a an odd integer in $\pi_1(\mathbb{R}P^1) \cong \mathbb{Z}$. This is a contradiction.

□

Corollary 9.6. *(Football squashing theorem) Every map $g : S^2 \longrightarrow \mathbb{R}^2$ identifies some pair of antipodal points.*

Proof. If there is a g which does not identify antipodal points, we always have $g(P) \neq g(-P)$, so we can define

$$f(P) := \frac{g(P) - g(-P)}{||g(P) - g(-P)||}$$

and obtain a map f contradicting the Borsuk-Ulam theorem. □

Example 9.7. *There is some pair of antipodal points on the earth with the same temperature and windspeed.*

Corollary 9.8. (Ham Sandwich Theorem) *If a ham sandwich is formed from two pieces of bread and one piece of ham then there is a single slice that will bisect all three.*

Proof. We suppose there is no such plane and reach a contradicition.

Define $f' : S^2 \longrightarrow \mathbb{R}^2 \setminus \{0\}$ as follows. A point $P \in S^2$ defines a perpendicular plane Π through the origin in $\mathbb{R}^3$. We translate Π in the P direction until it bisects the ham (if the ham has a gap, we take the middle plane parallel to Π bisecting the ham), and take

$$f'(P) = (Bread_1^+ - Bread_1^-, Bread_2^+ - Bread_2^-)$$

where $Bread_1^+$ is the weight of the first slice on the positive side of the plane and $Bread_1^-$ is the weight on the negative side, and similarly for the second slice. By assumption this does not take the value $(0, 0)$, so we may define

$$f : S^2 \longrightarrow S^1$$

by $f(P) = f'(P)/||f'(P)||$. We note that changing P to $-P$ changes the positive and negative sides of the plane so that $f(-P) = -f(P)$. Accordingly, f is a function of the type that the Borsuk-Ulam Theorem says cannot exist. □

Remark 9.9. Of course we did not prove f was continuous. This relies on some pretty reasonable assumptions about bread and ham.

9.2 The group operation and free group actions

To complete the picture, we should explain how the group operation works in terms of F.

Lemma 9.10. *With the hypotheses of the theorem, and using the bijection* $\ell : \pi_1(X, x_0) \xrightarrow{\cong} F$, *the group operation is as follows. Indeed, if* $\ell([\omega]) = \tilde{x}_1$, $\ell([\omega']) = \tilde{x}_1'$

$$\tilde{x}_1 \cdot \tilde{x}_1' = \ell_{\tilde{x}_1}([\omega']).$$

In words, we look for the end point $\tilde{\omega}(1) = \tilde{x}_1$ *of the lift* $\tilde{\omega}$ *of* ω *and then lift* ω' *from* $\tilde{x}_1$*, and look for the end of that path.* □

One very important example goes as follows. We suppose a discrete group G acts freely on $\tilde{X}$, and take $X = \tilde{X}/G$. Now consider the projection $p : \tilde{X} \longrightarrow X$

and note that if we pick a point $\tilde{x}_0 \in F := p^{-1}(x_0)$, then F is its orbit, so that the map $t_{x_0} : G \xrightarrow{\cong} F$ given by $t_{x_0}(g) = g\tilde{x}_0$ is a bijection.

Theorem 9.11. *Suppose G acts freely on $\tilde{X}$ and the projection map $p : \tilde{X} \longrightarrow \tilde{X}/G = X$ is a covering map (automatic if G is finite). If $\tilde{X}$ is path connected and $\pi_1(\tilde{X}, \tilde{x}_0)$ then we have a group isomorphism*

$$\pi_1(X, x_0) \cong G.$$

Proof. The map in question is the composite bijection

$$\pi_1(X, x_0) \xrightarrow[\ell]{\cong} F \xleftarrow[t_{\tilde{x}_0}]{\cong} G.$$

Suppose that we have two group elements g and g' and that

$$\ell([\omega]) = g\tilde{x}_0 = t_{\tilde{x}_0}(g) \text{ and } \ell([\omega']) = g'\tilde{x}_0. = t_{\tilde{x}_0}(g')$$

Thus $[\omega]$ corresponds to g and $[\omega']$ corresponds to g'.

In view of Lemma 9.10, we need only note that if $\tilde{\omega}'$ is the lift of ω' from $\tilde{x}_0$ then $g\tilde{\omega}'$ is a lift of ω' from $g\tilde{x}_0$ and $(g\tilde{\omega}')(1) = g(\tilde{\omega}(1))$. Thus

$$\ell([\omega] \cdot [\omega']) = \ell([\omega \cdot \omega']) = \ell_{g\tilde{x}_0}([\omega']) = (g\tilde{\omega}')(1) = g(\tilde{\omega}'(1)) = gg'\tilde{x}_0 = t_{\tilde{x}_0}(gg')$$

as required. □

Example 9.12. (The Klein bottle.) *Consider the two maps $A, B : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by $A(x, y) = (x + 1, y)$ and $B(x, y) = (-x, y + 1)$, and let $\Gamma = \langle A, B\rangle$ be the group of transformations of the plane generated by A and B. It is easy to see that*

$$ABA^{-1}B^{-1} = A^2$$

and in fact

$$\Gamma = \langle A, B \mid ABA^{-1}B^{-1} = A^2\rangle.$$

One may check that Γ acts freely on $\mathbb{R}^2$ and we write $K = \mathbb{R}^2/\Gamma$, noting that this is the Klein bottle. One may easily check that $\pi : \mathbb{R}^2 \longrightarrow K$ is a covering map and hence

$$\pi_1(K) \cong \Gamma.$$

Note that Γ is not abelian, so that K does not admit the structure of a topological group.