# 5 Homotopy and homotopy equivalence

This chapter is absolutely central to the course, and contains the core idea of algebraic topology.

More narrowly, the first purpose of this chapter is to consider the idea of when a map $f : X \longrightarrow Y$ can be deformed to a map $g : X \longrightarrow Y$. This is the notion of *homotopy*. We will then take this notion and use it to decide when a space $X$ can be deformed to a space $Y$. This is the notion of homotopy equivalence.

## 5.1 Homotopic maps

The idea is that we can move the first map to the second map.

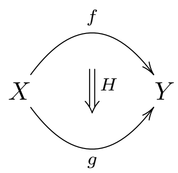
**Definition 5.1.** Let $X$, $Y$ be spaces and $f, g : X \longrightarrow Y$ continuous maps. We say that $f$ is *homotopic* to $g$ (and write $f \simeq g$) if there is a continuous map

$$H : X \times [0, 1] \longrightarrow Y$$

such that for all $x \in X$

$$H(x, 0) = f(x) \text{ and } H(x, 1) = g(x).$$

The map $H$ is then said to be a *homotopy* from $f$ to $g$ and write $H : f \simeq g$ or




We also write $H_t : X \longrightarrow Y$ for the map $H_t(x) = H(x, t)$. Thus $H_t$ gives a family of maps interpolating between $H_0 = f$ and $H_1 = g$. You may find it helpful to think of $t$ as time and the deformation actually starting with $f$ and moving to $g$.

You may also wish to note that for each $x \in X$ the homotopy $H$ gives a path $p_x : f(x) \longrightarrow g(x)$ in $Y$.

**Example 5.2.** This example makes precise the idea that the map from the circle to itself doing nothing is "more or less the same as" the map that rotates it by an angle $\phi$.

We have two maps $f, g : S^1 \longrightarrow S^1$ where $f$ is the identity and

$$g(e^{i\theta}) = e^{i(\theta+\phi)}.$$

We can now define a homotopy $H : f \Rightarrow g$ by

$$\begin{array}{rcccl} H & : & S^1 \times [0,1] & \longrightarrow & S^1 \\ & & (e^{i\theta}, t) & \mapsto & e^{i(\theta+t\phi)} \end{array}$$

**Example 5.3.** This example makes precise the idea that if we map the circle into $\mathbb{R}^2$ it's "more or less the same as" mapping it to a point, because we can shrink the circle down to a point in the plane. Again, this is simplest in polar coordinates.

We have two maps $f, g : S^1 \longrightarrow \mathbb{R}^2$ where $f$ is the inclusion and $g$ sends everything to the origin, i.e. for all $\theta$

$$\begin{array}{rcl} f(e^{i\theta}) & = & e^{i\theta} \\ g(e^{i\theta}) & = & (0,0) \end{array}$$

We can now define a homotopy $H : f \Rightarrow g$ by

$$\begin{array}{rcccl} H & : & S^1 \times [0,1] & \longrightarrow & \mathbb{R}^2 \\ & & (e^{i\theta}, t) & \mapsto & (1-t)e^{i\theta} \end{array}$$

## 5.2 Homotopy is an equivalence relation

We are going to show that homotopy is an equivalence relation on maps.

**Proposition 5.4.** *For any spaces $X$ and $Y$, homotopy is an equivalence relation on maps $X \longrightarrow Y$.*

**Proof.** We must show the relation is reflexive, symmetric and transitive.

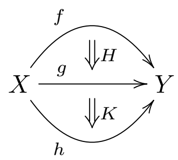
**Reflexive.** To see that $f \simeq f$ for any $f : X \longrightarrow Y$ we use the constant homotopy, given by

$$\begin{array}{ccc} X \times [0,1] & \longrightarrow & Y \\ (t, x) & \mapsto & f(x) \end{array}$$

**Symmetric.** If $f, g : X \longrightarrow Y$ and $f \simeq g$ we suppose $H$ is a homotopy from $f$ to $g$ and we use the "reverse homotopy" (exactly like the "reverse path"). We define the homotopy $\overline{H} : g \simeq f$ by

$$\begin{array}{rcccl} \overline{H} & : & X \times [0,1] & \longrightarrow & Y \\ & & (x,t) & \mapsto & H(x, 1-t) \end{array}$$

**Transitive.** If $f, g, h : X \longrightarrow Y$ with $H : f \simeq g$ and $K : g \simeq h$ we use the "concatenation" of the two homotopies (exactly like the concatenation of paths). We might depict the given data in the diagram




We define the concatenated homotopy by

$$\begin{array}{rcccl} H \cdot K & : & X \times [0,1] & \longrightarrow & Y \\ & & (x,t) & \mapsto & \begin{cases} H(x, 2t) & 0 \leq t \leq \frac{1}{2} \\ K(x, 2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases} \end{array}$$

This is continuous by the Gluing Lemma 2.15. □

We may then go on to show that composition of maps passes to homotopy classes.

**Lemma 5.5.** *Given maps $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$, there is a well defined composition of homotopy classes defined by*

$$[g] \circ [f] := [g \circ f].$$

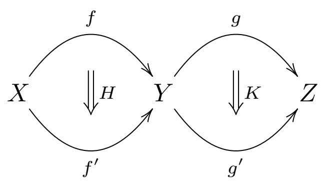
*This has the properties*

$$[f] \circ [id_X] = [f], [id_Y] \circ [f] = [f] \text{ and } ([h] \circ [g]) \circ [f] = [h] \circ ([g] \circ [f]).$$

**Proof.** The properties all follow immediately from those of functions once we know the result is well defined.

We must show that if $f \simeq f'$ and $g \simeq g'$ then $g \circ f \simeq g' \circ f'$.

In diagrammatic form, we are given



The homotopy $H$ gives us, for each $t \in [0,1]$ a continuous map

$$H_t : X \longrightarrow Y,$$

and $K$ gives us

$$K_t : Y \longrightarrow Z.$$

The idea is to compose these so that for each $t$ we have a map

$$K_t \circ H_t : X \longrightarrow Z,$$

and make this into a homotopy. Formally:

$$\begin{array}{rcccl} K \cdot H & : & X \times [0,1] & \longrightarrow & Z \\ & & (x,t) & \mapsto & K(H(x,t),t) \end{array}$$

□

## 5.3 Homotopy equivalence

Now that we have defined "homotopy" we are finally ready to make precise the idea of spaces being "more or less the same" given some squashing and stretching. The idea is that we want continuous maps between our two spaces that are not *exactly* inverse to one another, but only up to homotopy.

**Definition 5.6.** Two spaces $X$ and $Y$ are said to be *homotopy equivalent* (and we write $X \simeq Y$) if there are maps $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ such that

$$g \circ f \simeq id_X \text{ and } f \circ g \simeq id_Y$$

We then say that $f$ is a *homotopy equivalence*.

**Example 5.7.** *The space $X = \mathbb{R}^2 \setminus \{(0,0)\}$ is homotopy equivalent to $Y = S^1$. Indeed, we may define $f : X \longrightarrow Y$ by $f(x,y) = (x,y)/\sqrt{x^2+y^2}$, and $g(x,y) = (x,y)$.*

*Note that $f \circ g = 1_Y$ (which is then homotopic to $1_Y$ by using the constant homotopy). On the other hand, we may use the linear homotopy $H$ to see $g \circ f \simeq 1_X$. In other words*

$$H((x,y),t) = (1-t)g(f(x,y)) + t(x,y).$$

*Exactly similar arguments show*

$$\mathbb{R}^n \setminus \{0\} \simeq S^{n-1}$$

*for $n \geq 1$.*

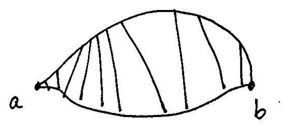
**Remark 5.8.** The sort of homotopy in this example is extremely useful, and so it is worth codifying a little.

Suppose we have maps $f, g : X \longrightarrow Y$ where $Y$ is a convex subset of a vector space. Then $f \simeq g$ via

$$\begin{array}{rcl} \alpha : I \times X & \longrightarrow & Y \\ (t, x) & \mapsto & (1-t).f(x) + t.g(x) \end{array}$$

We can check that $\alpha(0, x) = f(x)$ and $\alpha(1, x) = g(x)$. This is called a **straight line homotopy**. For example, any two paths in $\mathbb{R}^n$ are homotopic via a straight line homotopy. Note that if we have two paths with the same endpoints, i.e. $f, g : a \longrightarrow b \in \mathbb{R}^n$ then the straight line homotopy *keeps the endpoints fixed*:

$$\begin{array}{rclcl} \alpha(t, 0) & = & (1-t).a + t.a & = & a \\ \alpha(t, 1) & = & (1-t).b + t.b & = & b \end{array}$$




Note also that it isn't even essential that $Y$ is convex. It is only necessary that all the intervals $[f(x), g(x)]$ lie inside $Y$.

**Remark 5.9.** This is quite a common type of example. We have a space $X$ and a subspace $Y$, with $i : Y \longrightarrow X$ being inclusion, and a map $r : X \longrightarrow Y$ so that $i \circ r = 1_Y$. This much states that $Y$ is a *retract* of $X$.

The special feature is that in addition $r \circ i \simeq 1_X$, so that $X \simeq Y$. This sort of homotopy equivalence $r$ is said to be a *deformation retraction*, and we say that $Y$ is a *deformation retract* of $X$

**Proposition 5.10.** *Homotopy equivalence is an equivalence relation.*

**Proof.** Once again we must show reflexivity, symmetry and transitivity.

**Reflexive.** It is easy to see that $X \simeq X$, using the identity map in both directions and the constant homotopy.

**Symmetric.** If $X \simeq Y$ we have maps $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ whose composites in each direction are homotopic to the identity. Precisely the same information shows $Y \simeq X$.

**Transitive.** Suppose $X \simeq Y \simeq Z$ and that we have maps $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ with both composites homotopic to the identity and also $h : Y \longrightarrow Z$ and $k : Z \longrightarrow Y$ with both composites homotopic to the identity.

Then we use $h \circ f : X \longrightarrow Z$ and $g \circ k : Z \longrightarrow X$. Then we find

$$(h \circ f) \circ (g \circ k) = h \circ (f \circ g) \circ k \simeq h \circ id_Y \circ k = h \circ k \simeq id_Z$$

and

$$(g \circ k) \circ (h \circ f) = g \circ (k \circ h) \circ f \simeq g \circ id_Y \circ g = g \circ f \simeq id_X.$$

In both cases we have used Lemma 5.5 to see homotopies can be composed. □

The equivalence classes under homotopy equivelnce are called *homotopy types.*

## 5.4 $\pi_0$ is homotopy invariant

In fact all the main invariants $I$ defined in this course will be invariant under homotopy equivalence, in the sense that if $X \simeq Y$ then $I(X) \cong I(Y)$. Usually one can be a bit more precise. One may say that a particular homotopy equivalence $f : X \xrightarrow{\simeq} Y$ induces an isomorphsm $f_* : I(X) \xrightarrow{\cong} I(Y)$.

Indeed, for any continuous function $f : X \longrightarrow Y$ we have an induced map

$$f_* : \pi_0(X) \longrightarrow \pi_0(Y)$$

defined by

$$f_*([x]) := [f(x)].$$

To see this is well defined we note that if $\gamma$ is a path from $a$ to $b$ in $X$ (so that $[a] = [b]$) then $f\gamma$ is a path from $f(a)$ to $f(b)$ in $Y$ (so that $[f(a)] = [f(b)]$).

**Lemma 5.11.** *The above definition has the properties*

- $(id_X)_* = id_{\pi_0(X)}$
- $(gf)_* = g_* f_*$.
- *If* $f \simeq f'$ *then* $f_* = (f')_*$

**Proof.** The first property is immediate. For the second, we calculate

$$(gf)_*([x]) = [(gf)(x)] = [g(f(x))] = g_*[f(x)] = g_*(f_*[x]) = (g_* f_*)([x]).$$

Finally if $H : f \simeq f'$ we have seen that restricting $H$ to $\{x\} \times [0,1]$ gives a path from $f(x)$ to $f'(x)$ so that

$$f_*([x]) = [f(x)] = [f'(x)] = (f')_*([x])$$

as required. □

**Remark 5.12.** This shows that $\pi_0$ is an example of a *homotopy invariant functor*

$$\pi_0 : \text{Top} \longrightarrow \text{Sets}.$$

It is worth explaining this in a little more detail.

## 5.5 Categories and homotopy invariant functors

To explain, recall that a *category* consists of a class of objects (like sets, groups or topological spaces) and between any two objects $X, Y$ of this type there is a class of *morphisms* $f : X \longrightarrow Y$ (functions, group homomorphisms, continuous functions in the above examples). The only axioms are that for each object $X$ there is an *identity morphism* $id_X : X \longrightarrow X$ and that there is an associative composition. We will probably get by with four categories.

- The category Sets of sets and functions
- The category Top of topological spaces and continuous functions
- The category Groups of groups and group homomorphisms
- The category AbGrps of abelian groups and group homomorphisms

This lets us explain that $\pi_0$ is an example of a construction

$$I : \text{Top} \longrightarrow \mathbb{C}$$

taking a space $X$ to an object $I(X)$ of a category $\mathbb{C}$ and a continuous function $f : X \longrightarrow Y$ to a map $f_* : I(X) \longrightarrow I(Y)$ of objects of $\mathbb{C}$.

It has the properties

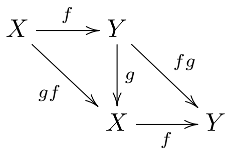
- $(id_X)_* = id_{I(X)}$
- $(gf)_* = g_* f_*$.
- If $f \simeq f'$ then $f_* = (f')_*$

The first two properties are called *functoriality* and we say that $I$ is a *functor*. The third says that $I$ is invariant under homotopy.

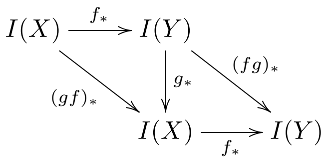
The following elementary observation could be said to be the central idea of algebraic topology.

**Lemma 5.13.** *A homotopy invariant functor gives an invariant of homotopy type.*

**Proof.** If $X \simeq Y$ then we have maps




Applying $I$ we get the diagram




Now use the fact that since $gf = id_X$ we have

$$g_* f_* = (gf)_* = (id_X)_* = id_{I(X)}$$

and similarly $f_* g_* = id_{I(Y)}$, so that $f_*$ and $g_*$ give inverse isomorphisms showing

$$I(X) \cong I(Y).$$

Thus functoriality and homotopy show that $I$ is a homotopy type invariant. $\square$

## 5.6 Contractions and deformation retractions

In fact there is a special case of this which is extremely important.

**Definition 5.14.** A space is called **contractible** if it has the homotopy type of a point, that is, it is homotopy equivalent to a point.

**Lemma 5.15.** *A space $X$ is contractible if and only if there is a homotopy $\mathbf{c}_a \simeq \mathbf{id}_X$ for some point $a \in X$. Here $\mathbf{c}_a$ is the map $X \longrightarrow X$ that sends everything to $a$.*

**Proof.** $X$ is contractible if and only if we have maps $f : X \longrightarrow *$ and $g : * \longrightarrow X$ together with homotopies $gf \simeq \mathbf{id}_X$ and $fg \simeq \mathbf{id}_*$.

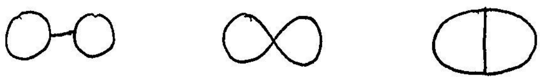
Now $fg = id_*$ so the second homotopy can be taken to be the identity. For the first homotopy, first observe that a map $g : * \longrightarrow X$ simply picks out a point in $X$, so the composite $gf$ sends everything to this point $g(*)$. So it is $\mathbf{c}_a$, where we write $a = g(*)$. Thus a homotopy $gf \simeq \mathbf{id}_X$ is just a homotopy $\mathbf{c}_a \simeq \mathbf{id}_X$, hence the result. □

**Example 5.16.** $\mathbb{R}^2$ is contractible. We take $a$ to be the origin and construct a homotopy $\mathbf{id}_{\mathbb{R}^2} \simeq \mathbf{c}_a$ using the linear homotopy

$$\begin{array}{ccc} \mathbb{R}^2 \times [0,1] & \longrightarrow & \mathbb{R}^2 \\ ((x,y),t) & \mapsto & ((1-t)x,(1-t)y) \end{array}$$

With luck the following examples will now look natural, though it might take a fair amount of paper to write down full details.

**Example 5.17.** The following spaces are homotopy equivalent:



So are the following spaces:

**Exercise 5.18.** Show that the following two spaces are homotopy equivalent:

# 6 The fundamental group

## 6.1 Path homotopy

We have discussed paths in a space as ways of getting from one point to another. We have also suggested that some routes from $a$ to $b$ are more different than others. We are now going to make that precise.

**Definition 6.1.** Let $\omega, \sigma : [0,1] \longrightarrow X$ be paths from $a$ to $b$ in $X$. Then $\omega$ and $\sigma$ are **homotopic** if they are homotopic through paths from $a$ to $b$. That is, we have a map $H : [0,1] \times [0,1] \longrightarrow X$ such that

$$\begin{aligned} H(s,0) &= \omega(s) \\ H(s,1) &= \sigma(s) \end{aligned}$$

for all $s \in [0,1]$ and

$$\begin{aligned} H(0,t) &= a \\ H(1,t) &= b \end{aligned}$$

for all $t \in [0,1]$.

The map $H$ is called a **path homotopy**. Similarly, if both the paths are loops, we may refer to **loop homotopy**.

**Example 6.2.** *(a) Any two paths $\omega, \sigma$ between the same points in $\mathbb{R}^n$ are homotopic using the linear homotopy $H(s,t) = (1-t)\omega(s) + t\sigma(s)$.*

*(b) If $l_\theta : [0,1] \longrightarrow S^2$ is the uniform path from the north pole to the south pole along the $\theta$ line of longditude, then $l_\theta \simeq l_\phi$ for any $\theta$ and $\phi$. Indeed, we may take $H_t = l_{(1-t)\theta+t\phi}$.*

Exactly as for ordinary homotopies, path homotopy is a very good relation.

**Lemma 6.3.** *Path homotopy is an equivalence relation on paths from $a$ to $b$ in $X$.*

**Proof.** Exercise (See Proposition 5.4). □

We have already defined the constant path, the reverse path and the concatenation of paths. We should check they are compatible with homotopy.

**Lemma 6.4.** *(a) If $\omega, \omega'$ are paths from $a$ to $b$ in $X$ and $\omega \simeq \omega'$ then $\overline{\omega} \simeq \overline{\omega'}$.*

*(b) If in addition $\sigma, \sigma'$ are paths from $b$ to $c$ in $X$ and $\sigma \simeq \sigma'$ then $\omega \cdot \sigma \simeq \omega' \cdot \sigma'$.*

**Proof.** We will prove Part (b). Suppose $H : \omega \simeq \omega'$ and $K : \sigma \simeq \sigma'$, then we may define $H \cdot K$ by

$$(H \cdot K)(s,t) = \begin{cases} H(s, 2t) \text{ for } t \in [0, 1/2] \\ K(s, 2t-1) \text{ for } t \in [1/2, 1]. \end{cases}$$

This gives a homotopy $\omega \cdot \sigma \simeq \omega' \cdot \sigma'$ as required. □

## 6.2 The definition

The fundamental group consists of loops up to homotopy.

**Definition 6.5.** If $X$ is a topological space and $x_0$ is a point of $X$ (called the *basepoint*) the *fundamental group* is defined to be the set of path homotopy classes of loops based at $x_0$

$$\pi_1(X, x_0) = \{\omega \mid \omega \text{ is a loop based at } x_0\ \}/ \simeq .$$

We have not yet justified the word 'group' in this definition.

**Theorem 6.6.** *The set $\pi_1(X, x_0)$ is a group under the operation*

$$[\omega] \cdot [\sigma] = [\omega \cdot \sigma].$$

*The identity element is $[c_{x_0}]$. The inverse operation is*

$$[\omega]^{-1} = [\overline{\omega}].$$

**Proof.** By Lemma 6.4 the group operation and the inverse operation are well defined. It is clear that if $\omega$ and $\sigma$ are loops based at $x_0$, so is their concatenation $\omega \cdot \sigma$, so the binary operation is closed.

It remains to show that the binary operation is associative, has an identity and has inverses.

*Associativity:* By definition, given three loops $\omega, \sigma, \tau$, we have

$$([\omega] \cdot [\sigma]) \cdot [\tau] = [\omega \cdot \sigma] \cdot [\tau] = [(\omega \cdot \sigma) \cdot \tau]$$

and

$$[\omega] \cdot ([\sigma]) \cdot [\tau]) = [\omega] \cdot [\sigma \cdot \tau] = [\omega \cdot (\sigma \cdot \tau)].$$

It therefore remains to write down a homotopy

$$(\omega \cdot \sigma) \cdot \tau \simeq \omega \cdot (\sigma \cdot \tau).$$

We draw a picture. Exercise: write out formulae.

*Identity:* We must check that

$$[\omega] \cdot [c_{x_0}] = [\omega \cdot c_{x_0}] = [\omega] = [c_{x_0} \cdot \omega] = [c_{x_0}] \cdot [\sigma].$$

It therefore suffices to wrtie down homotopies

$$c_{x_0} \cdot \omega \simeq \omega \simeq c_{x_0} \cdot \sigma.$$

Again, we draw a picture. In symbols

$$H(s,t) = \begin{cases} \omega(\frac{2t}{1+s}) \text{ for } t \in [0, \frac{1+s}{2}] \\ x_0 \text{ for } t \in [\frac{1+s}{2}, 1] \end{cases}$$

*Inverse:* We must show

$$[\omega] \cdot [\omega]^{-1} = [c_{x_0}] = [\omega]^{-1}[\omega].$$

Since our inverse operation is an involution, it suffices to prove the first. Since

$$[\omega] \cdot [\omega]^{-1} = [\omega \cdot \overline{\omega}]$$

it suffices to construct a homotopy $\omega \cdot \overline{\omega} \simeq c_{x_0}$ We define

$$H(s,t) = \begin{cases} \omega(2s) \text{ if } s \in [0, t/2] \\ \omega(t) \text{ if } s \in [t/2, 1 - t/2] \\ \overline{\omega}(2s - 1) \text{ if } s \in [1 - t/2, 1] \end{cases}$$

□

## 6.3 Functoriality

We would like to know that the fundamental group is an invariant of the space $X$. The first step is to see how it behaves under continuous functions.

**Definition 6.7.** *If $f : X \longrightarrow Y$ is a continuous function then we may define the induced map*

$$f_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, f(x_0)).$$

*by $f_*([\omega] = [f \circ \omega]$.*

**Remark 6.8.** We note that this is well defined by Lemma 5.5 since if $\omega \simeq \omega'$ then $f \circ \omega \simeq f \circ \omega'$.

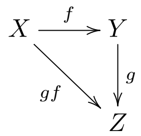
**Lemma 6.9.** *The induced map $f_*$ is a group homomorphism.*

*Proof.* We must show $f_*([\omega] \cdot [\sigma]) = f_*([\omega]) \cdot f_*([\sigma])$. In other words, we need to give a path homotopy $f \circ (\omega \cdot \sigma) \simeq (f \circ \omega) \cdot (f \circ \sigma)$. However, since the two are actually *equal* we may just use the identity homotopy. □

**Lemma 6.10.** *The induces map is functorial in the sense that*

$$(id_X)_* = id_{\pi_1(X,x_0)}$$

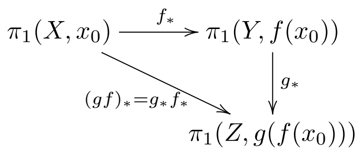
*and given*




*we have*

$$(g \circ f)_* = g_* \circ f_*.$$

*Diagrammatically we have*




**Remark 6.11.** This means that $\pi_1$ is a functor from based spaces (i.e., the category $\mathbf{Top}_*$ whose objects are spaces with a specified basepoint, and whose morphisms are continuous functions taking the basepoint to the basepoint) to groups.

*Proof.* Knowing the induced map is well defined, both statements are straightforward. The one about the identity is immediate. For composites, we note that for any loop $\omega$ we have

$$(gf)_*[\omega] = [(g \circ f) \circ \omega] = [g \circ (f \circ \omega)] = g_*[f \circ \omega] = g_*(f_*[\omega]) = (g_* \circ f_*)([\omega]).$$

□

Finally, we need to be a little careful for homotopies.

**Lemma 6.12.** *If $f, f' : X \longrightarrow Y$ are two maps taking $x_0$ to $y_0$ and there is a homotopy $H : f \simeq f'$ which does not move $x_0$ then*

$$f_* = (f')_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0).$$

*Proof.* We need only observe that we may compose $H : X \times [0,1] \longrightarrow Y$ with $\omega$ to obtain a homotopy

$$[0,1] \times [0,1] \overset{\omega \times id}{\longrightarrow} X \times [0,1] \overset{H}{\longrightarrow} Y.$$

By the hypothesis on $H$, this is a path homotopy $f \circ \omega \simeq f' \circ \omega$ and

$$f_*[\omega] = [f \circ \omega] = [(f') \circ \omega] = (f')_*[\omega].$$

□

By the discussion above, we conclude that if $X \simeq Y$ is a homotopy equivalence in which the homotopies preserve the basepoint, then $\pi_1(X, x_0) \cong \pi_1(Y, y_0)$.

**Example 6.13.** *We see that if $C$ is any convex subset of $\mathbb{R}^n$ and $x_0 \in C$ is any point then $\pi_1(C, x_0)$ is trivial. Furthermore if $A$ is a retract of a space $C$ with trivial fundamental group $\pi_1(C, a_0)$ then $\pi_1(A, a_0)$ is trivial.*

**Remark 6.14.** In fact if $f \simeq f'$ by a homotopy in which $y_0 = f(x_0)$ moves to $y_0' = f'(x_0)$ along a path $\sigma$ then there is an isomorphism $\tau_\sigma$ (see Exercises 2) so that

$$\begin{array}{ccc} & & \pi_1(Y, f(x_0)) \\ & \overset{f_*}{\nearrow} & \\ \pi_1(X, x_0) & & \cong \Big\downarrow t_\omega \\ & \underset{(f')_*}{\searrow} & \\ & & \pi_1(Y, (f')(x_0)) \end{array}$$

commutes.