16 The simplicial approximation theorem

The purpose of this section is to show that any continuous function $f : X \longrightarrow Y$ between triangulable spaces is approximated by a simplicial map. More precisely if we have triangulations $t_X : |K| \stackrel{\cong}{\longrightarrow} X$ and $t_Y : |L| \stackrel{\cong}{\longrightarrow} Y$, we can subdivide K enough and then find a simplicial map $s : K' \longrightarrow L$ approximating f at least in the sense that $|s| \simeq f$. We will need to explain about subdivision and approximation.

16.1 Subdivision

The idea of subdivision is to make the simplices smaller and smaller. So it is really essential to be talking about *geometric* simplicial complexes.

Definition 16.1. *(a) If K is a simplicial complex (geometric or abstract) then a k-flag in K is*

$$(\sigma_0 \supset \sigma_1 \supset \cdots \supset \sigma_k)$$

where $\sigma_i \in K$ for all i. T

(b) The barycentric subdivision *of an abstract simplicial complex K is the abstract simplicial complex with $V(K') = K$ and K' consisting of the flags of K.*

(b) If K is an geometic simplicial complex then we form the barycentric subdivision *of K is the goemetric simplicial complex with $V(K') = K$ and K' has simplices*

$$\langle \hat{\sigma}_0, \hat{\sigma}_1, \ldots, \hat{\sigma}_k \rangle$$

where

$$(\sigma_0 \supset \sigma_1 \supset \cdots \supset \sigma_k)$$

is a flag in K.

The first point is that subdivision has just chopped up the same subspace of $\mathbb{R}^n$ into smaller simplices.

Lemma 16.2. *If K is a geometric simplicial complex then the geometric realizations of K and K' are equal:*

$$|K'| = |K|.$$

Proof. It suffices to deal with an individual simplex $\sigma = \langle v_0, \ldots, v_k \rangle$.

For $P \in |\sigma|$, consider barycentric coordinates $P = \sum_i \lambda_i v_i$. Suppose that the set of different values λ_i has $k+1$ elements, $\mu_0 > \mu_1 > \cdots > \mu_k$, and let

$$\sigma_j = \{v_i \mid \lambda_i \geq \mu_j\}.$$

Thus

$$(\sigma_0 \subset \sigma_1 \subset \cdots \subset \sigma_k).$$

We then find

$$P \in \langle \hat{\sigma}_0, \hat{\sigma}_1, \ldots, \hat{\sigma}_k \rangle,$$

and that P does not lie in any proper face. □

The point of K' is that $|K'| = |K|$ but the simplices of K' are smaller.

Definition 16.3. *The* mesh *of a geometric simplicial complex is*

$$\text{mesh}(K) = \max\{\text{diam}(\sigma) \mid \sigma \in K\}$$

We should give an estimate for how much smaller the simplices get under subdivision.

Lemma 16.4. *If K is of dimension d then*

$$\text{mesh}(K') \leq \frac{d}{d+1}\text{mesh}(K).$$

Proof. A typical edge of K' is of the form $(\sigma \supset \tau)$. The largest mesh has $\tau = v_0$ for some vertex v_0 of σ. Then

$$\text{diam}(\langle \hat{\sigma}, v_0 \rangle) \leq \frac{k}{k+1}\text{diam}(\sigma) \leq \frac{d}{d+1}\text{diam}(\sigma) \leq \frac{d}{d+1}\text{mesh}(K)$$

□

Corollary 16.5. *For any $\epsilon > 0$ there is an n so that* $\text{mesh}(K^{(n)}) < \epsilon$.

16.2 Simplicial approximation

The idea of simplicial approximation is that every element of $|K|$ belongs to a particular simplex, and an approximation of $f : |K| \longrightarrow |L|$ is another map g for which so that $f(x)$ and $g(x)$ always belong to the same simplex in this sense.

Definition 16.6. *(a) For $x \in K$,* $\text{carrier}_K(x)$ *is the unique simplex σ with $x \in \sigma^{so} = |\sigma| \setminus \bigcup_{\tau \subset \sigma} |\tau|$.*

(b) If $f : |K| \longrightarrow |L|$ then $s : K \longrightarrow L$ is a simplicial approximation *to f if $|s|(x) \in \text{carrier}_L(f(x))$ for all $x \in |K|$.*

We should relate this form of approximation to the idea of deformation we've seen before.

Lemma 16.7. *If s is a simplicial approximation of f then $f \simeq |s|$.*

Proof. We may use the linear homotopy

$$H : |K| \times [0,1] \longrightarrow |L|$$

defined by

$$H(x,t) = t|s|(x) + (1-t)f(x)$$

because both $|s|(x)$ and $f(x)$ are both in the same simplex. □

The main theorem, and the main reason that simplicial homology can give a topological invariant, is that any continuous function can always be approximated by a simplicial map.

Theorem 16.8. *If $f : |K| \longrightarrow |L|$ is continuous then there is an m so that there is a simplicial approximation*

$$s : K^{(m)} \longrightarrow L$$

to f.

Even at the crudest level this transforms the landscape. Up to homotopy there are no space-filling curves.

Corollary 16.9. *If K is d-dimensional then any continuous function $f : |K| \longrightarrow |L|$ is homotopic to a function into the d-skeleton of L.* □

Even for continuous functions, triangulable spaces have strong finiteness properties.

Corollary 16.10. *There are only countably many homotopy classes of maps $|K| \longrightarrow |L|$ for any simplicial complexes K and L.*

We may ensure that simplicial approximation does not move the values of the function very much.

Remark 16.11. Given $\epsilon > 0$ we may even subdivide L first and ensure that there is a simplicial approximation $t : K^{(m)} \longrightarrow L^{(n)}$ so that

$$d(f(x), |t|(x)) < \epsilon \text{ for all } x \in |K|$$

16.3 Proof of the Simplicial Approximation Theorem

To connect the combinatorics to the topology we will need some combinatorially defined open sets.

Definition 16.12. *If u is a vertex of K then the open star of u in K is*

$$\mathrm{star}^\circ_K(u) = \bigcup_{u\in\sigma\in K} \sigma^{so}.$$

The simplices can be recovered from the open stars.

Lemma 16.13.

$$\{v_0,\dots,v_k\} \Longleftrightarrow \bigcap_{i=0}^{k} \mathrm{star}^\circ_K(v_i) \neq \emptyset. \quad \square$$

We are ready to prove the Simplicial Approximation Theorem.

Proof. Consider the open cover

$$\mathcal{V} = \{\mathrm{star}^\circ_L(v) \mid v \text{ a vertex of } L\}$$

of $|L|$. Since f is continuous this gives an open cover

$$\mathcal{U} = \{f^{-1}(\mathrm{star}^\circ_L(v)) \mid v \text{ a vertex of } L\}$$

of $|K|$. Now $|K|$ is compact so we may take a Lebesgue number $\delta > 0$ for the cover $\mathcal{U}$ and choose m so large that $\mathrm{mesh}(K^{(m)}) \leq \delta/2$.

Thus for each vertex u of K we find $\mathrm{star}^\circ_K(u)$ (being of diameter $\leq \delta$) lies in some element $f^{-1}(\mathrm{star}^\circ_L(v))$ of $\mathcal{U}$. In other words, there is a $v = v(u)$ so that

$$f(\mathrm{star}^\circ_K(u)) \subseteq \mathrm{star}^\circ_L(v).$$

We define

$$s(u) = v$$

and note that by Lemma 16.13 it is a simplicial map $s : K^{(m)} \longrightarrow L$.

Finally we must show that s is a simplicial approximation to f. So we suppose $x \in |K|$ and $x \in \langle U - 0, \dots, u_k\rangle$. Thus

$$s(x) \in \langle s(u_0), \dots, s(u_k)\rangle \subseteq \mathrm{carrier}_L(f(x)),$$

as required. $\square$

17 Topological invariance of simplicial homology

In this section we prove that homology gives a homotopy invariant of triangulated spaces.

Theorem ∞: We may associate to any continuous function $f : |K| \longrightarrow |L|$ an induced map $f_* : H_*(K) \longrightarrow H_*(L)$ in such a way that

- $(id_{|K|})_* = id_{H_*(K)}$
- $(g \circ f)_* = g_* \circ f_*$
- $f \simeq f' \Rightarrow f_* = (f')_*$.

This immediately shows that homotopy equivalences induce homology isomorphisms and therefore we have a homotopy type invariant.

Corollary 17.1. *If* $|K_1| \simeq |K_2|$ *then* $H_*(K_1) \cong H_*(K_2)$.

We will sketch here how this is proved. Given $f : |K| \longrightarrow |L|$ is continuous, we define $f_* : H_*(K) \longrightarrow H_*(L)$ as follows.

By the Simplicial Approximation Theorem 16.8 there is an integer m and a simplicial map
$$s : K^{(m)} \longrightarrow L$$
which a simplicial approximation to f in the sense that $|s|(x) \in \text{carrier}_L(f(x))$ for all $x \in |K|$.

Definition 17.2. *We define* $f_*^{m,s}$ *to be the composite*
$$H_*(K) \xrightarrow[\cong]{\text{Sbd}^m} H_*(K^{(m)}) \xrightarrow{s_*} H_*(L),$$
where Sbd *is the map induced by chopping up simplices.*

It remains to show

- *Sbd* is an isomorphism in homology
- The homomorphism $f_*^{m,s}$ is independent of m and s.
- f_* (now known to be independent of m and s is functorial in the sense that $id_* = id$ and $(g \circ f)_* = g_* \circ f_*$.
- $f \simeq f'$ implies $f_* = (f')_*$.

17.1 Subdivision

Recall that for a simplical complex K the barycentric subdivision K' consists of chains of simplices of K. In the geometric setting this means that if

$$\sigma_0 \supset \sigma_1 \supset \cdots \supset \sigma_k \text{ is a chain of simplices in } K$$

then

$$\langle \hat{\sigma}_0, \hat{\sigma}_1, \ldots, \hat{\sigma}_k \rangle \text{ is a simplex of } K'.$$

Definition 17.3. *Barycentric subdivision*

$$\mathrm{Sbd} : C_k(K) \longrightarrow C_k(K')$$

is defined by the formula

$$\mathrm{Sbd}(\langle v_0, v_1, \ldots, v_k \rangle) = \sum_{\pi} (\pi) \langle \langle v_{\pi 0}, \hat{\rangle}, \langle v_0, v_1 \hat{\rangle}, \ldots, \langle v_{\pi 0}, v_{\pi 1}, \ldots, v_{\pi k} \hat{\rangle} \rangle$$

One may check directly that this is a chain map, but in fact we will do this by breaking it down into many steps each of which is a chain map.

A picture is more effective than the following definition.

Definition 17.4. *If A is an n-simplex we take the A-*stellar subdivision *of K to be defined by*

$$\mathrm{St}_A(K) = \{\sigma \in K \mid A \not\subseteq \sigma\} \cup \{\tau \cup \{\hat{A}\} \mid A \not\subseteq \tau, A \cup \tau \in K\}$$

It is then apparent that if we do stellar subdivision in order of increasing dimension, we eventually reach the barycentric subdivision.

Lemma 17.5. *If we list the simplices of K as $A_1, A_2, \ldots, A_N$ so that* $\dim(A_i) \leq \dim(A_{i+1})$ *then*

$$K' = \mathrm{St}_{A_1} \mathrm{St}_{A_2} \cdots \mathrm{St}_{A_N} K. \quad \square$$

Once again we can define subdivision at the level of chain complexes by adding up the simplices in the subdivision. Now the formula becomes easier to manage.

Definition 17.6. *If $A = \langle a_0, \ldots, a_n \rangle$ then subsdivision at A*

$$\mathrm{Sbd}_A : C_k(K) \longrightarrow C_k(K')$$

is defined by taking

$$\mathrm{Sbd}_A(\sigma) = \sigma \text{ if } A \not\subseteq \sigma$$

and

$$\mathrm{Sbd}_A(\langle a_0, a_1, \ldots a_n, v_{n+1}, \ldots, v_k \rangle) = \sum_i (-1)^i \langle \hat{A}, a_0, \ldots, \hat{a_i}, \ldots, a_n, v_{n+1}, \ldots, v_k \rangle$$

Lemma 17.7. Sbd_A *is a chain map.*

Proof. Calculate $d(\mathrm{Sbd}_A(\sigma))$ and $\mathrm{Sbd}_A(d(\sigma))$ for a simplex σ, and observe they are equal. □

Theorem 17.8. *Subdivision induces an isomorphism in homology:*

$$(\mathrm{Sbd}_A)_* : H_*(K) \xrightarrow{\cong} H_*(\mathrm{St}_A(K)).$$

Proof. In fact we define a homology inverse to the chain map $\mathrm{Sbd}_A : C_k(K) \longrightarrow C_k(K')$. This map involves the choice of a vertex a_0 of A, but then it is easy: we simply use the simplicial map

$$\theta : \mathrm{St}_A K \longrightarrow K$$

defined by

$$\theta(v) = v \text{ if } v \text{ is a vertex of } K \text{ and } \theta(\hat{A}) = a_o.$$

To see it is simplical note that if τ does not involve $\hat{A}$ then $\theta(\tau) = \tau \in K$ whilst if $\tau = \sigma \cup \{\hat{A}\}$ then $A \not\subseteq \sigma$ and $A \cup \sigma \in K$ so that $\{a_0\} \cup \sigma \in K$.

We now just verify

- $\theta_* \circ (\mathrm{Sbd}_A)_* = id$
- $(\mathrm{Sbd}_A)_* \circ \theta_* = id.$

The first even holds at the chain level: $\theta_\sharp \circ \mathrm{Sbd}_A = id$. Indeed if σ is a simplex not containing A then

$$\theta_\sharp(\mathrm{Sbd}_A(\sigma)) = \theta_\sharp(\sigma) = \sigma.$$

On simplices containing A, the definition states

$$\mathrm{Sbd}_A(\langle a_0, a_1, \dots a_n, v_{n+1}, \dots, v_k\rangle) = \sum_i (-1)^i \langle \hat{A}, a_0, \dots, \hat{a}_i, \dots, a_n, v_{n+1}, \dots, v_k\rangle.$$

Applying θ replaces $\hat{A}$ by a_0. This gives zero except in the term with $i = 0$ when we recover the original simplex.

Finally, we verify $(\mathrm{Sbd}_A)_* \circ \theta_* = id$. Suppose we have a homology class $\alpha \in H_k(\mathrm{St}_A K)$ and let $z \in Z_k \mathrm{St}_A K$ be a representative cylce. We will show that $e = z - \mathrm{Sbd}_A(\theta_\sharp(z))$ is a boundary and conclude

$$\alpha = [z] = [z - e] = [\mathrm{Sbd}_A(\theta_\sharp(z))] = \mathrm{Sbd}_{A*}(\theta_*(\alpha))$$

as required.

Now observe that e is a sum of simplices involving $\hat{A}$ since Sbd_A and $\theta_\sharp$ are the identity elsewhere. Now let

$$L_0 = \{\sigma \in \mathrm{St}_A K \mid \hat{A} \in \sigma\}$$

$$L = \{\tau \mid \tau \subseteq \sigma \in L_0\} \text{ and } M := \mathrm{link}_{\mathrm{St}_A K}(\hat{A}) := L \setminus L_0,$$

and observe that L is a cone:

$$L = c_{\hat{A}}(M).$$

Now e is a chain on L and $H_k(c_{\hat{A}}M) = 0$ by Proposition 13.6, so that to verify that e is a boundary we need only observe that since $e = z - \mathrm{Sbd}_A(\theta_\sharp(z))$ it is obviously a cycle. □

Corollary 17.9. *Barycentric subdivision is also an isomorphism*

$$\mathrm{Sbd} : H_*(K) \xrightarrow{\cong} H_*(K').$$

17.2 The induced map

We next show that given $f : K \longrightarrow L$ there is a well defined induced map. More precisely, if we choose m and a simplicial approximation $s : K^{(m)} \longrightarrow L$ then the map $f_*^{m,s}$ defined as the composite

$$H_*(K) \xrightarrow[\cong]{\mathrm{Sbd}^m} H_*(K^{(m)}) \xrightarrow{s_*} H_*(L)$$

is independent of m and s.

We will in effect being using a combinatorial counterpart of homotopy.

Definition 17.10. *Two simplicial maps $s, t : K \longrightarrow L$ are* close *if, for any simplex $\sigma \in K$, there is a simplex $\tau \in L$ so that $s(\sigma)$ and $t(\sigma)$ are faces of τ.*

We immediately have examples of closeness.

Lemma 17.11. *If $s, t : K \longrightarrow L$ are both simplicial approximations to $f : |K| \longrightarrow |L|$ then s and t are close.*

Proof. If $\sigma = \{u_0, \ldots, u_k\}$ then

$$f(\hat{\sigma}) = f(\frac{1}{k+1}\sum_i u_i) \in \mathrm{star}^\circ(s(\frac{1}{k+1}\sum_i u_i)) \cap t(\frac{1}{k+1}\sum_i u_i)$$
$$= \bigcap_i \mathrm{star}^\circ(s(u_i) \cap \mathrm{star}^\circ(t(u_i))$$

Hence $\{s(u_0), , s(u_k), t(u_0), \ldots, t(u_k)\}$ is a simplex. □

The power of closeness is the implications for homology.

Lemma 17.12. *If* $s, t : K \longrightarrow L$ *are close then* $s_\sharp \simeq t_\sharp$ *and hence*

$$s_* = t_* : H_*(K) \longrightarrow H_*(L).$$

Proof. We define a chain homotopy inductively. □

Corollary 17.13. *The map* $f_*^{m,s}$ *is independent of* m, s

Proof. We suppose $s : K^{(m)} \longrightarrow L$ and $t : K^{(n)} \longrightarrow L$ are two simplicial approximations to f. Suppoe $n = m + r$ with $r \geq 0$. We claim that the composite

$$K^{(m+r)} \xrightarrow{\theta} K^{(m)} \xrightarrow{s} L$$

is also a simplicial approximation to f. By Lemma 17.12 it follows that $s \circ \theta$ and t induce the same map in homology.

To see that $s \circ \theta$ is a simplicial approximation to f, note that for $x \in |K|$,

$$s(x) \in \text{carrier}_K(f(x)) \Rightarrow s(\text{carrier}_{K^{(m)}}(x)) \subseteq \text{carrier}_K(f(x))$$

However $\text{carrier}_{K^{(m)}}(\theta(x))$ has vertices from amongst $\text{carrier}_{K^{(m+r)}}(x)$, so

$$\text{carrier}_{K^{(m+r)}}(x) \subseteq \text{carrier}_{K^{(m+r)}}(x)$$

and

$$|s \circ \theta|(x) \in |s|(\text{carrier}\theta(x)) \subseteq |s|(\text{carrier}(x)) \subseteq |(f(x))|$$

as required. □

17.3 Functoriality

If is immediate that $id_* = id$, since the identity simplicial map is a a simplicial approximation to the identtity

Suppose than that we have

$$|K| \xrightarrow{f} |L| \xrightarrow{g} |M|$$

with specified simplicial approximations

$$K^{(m)} \xrightarrow{s} L^{(n)} \xrightarrow{t} M$$

(i.e., we first approximate g giving n and t, and then approximate f, viewed as a map $|K| \longrightarrow |L^{(n)}| = |L|$.

Now we have a diagram

$$\begin{array}{ccccc} H_*(K) & \xrightarrow{f_*} & H_*(L) & & \\ \mathrm{Sbd}_*^m \downarrow \cong & & \mathrm{Sbd}_*^n \updownarrow \cong & \searrow^{g_*} & \\ H_*(K^{(m)}) & \xrightarrow{s_*} & H_*(L^{(n)})\theta & \xrightarrow{t_*} & H_*(M) \end{array}$$

Hence

$$\begin{aligned} g_* \circ f_* &= g_*^{n,t}{}_*^{m,\theta\circ s} \\ &= (t_*\mathrm{Sbd}_*^n) \circ (s_*\theta_*\mathrm{Sbd}_*^m) \\ &= t_*s_*\mathrm{Sbd}_*^m \\ &= (t \circ s)_*\mathrm{Sbd}_*^m \\ &= (g \circ f)_* \end{aligned}$$

where the first equality is because θs approximates f and the last is because ts approximates gf.

17.4 Homotopy

Suppose $f \simeq f' : |K| \longrightarrow |L|$.

We will show there is an m and a k and simplicial maps $s_0, s_1, \ldots, s_{k-1} : K^{(m)} \longrightarrow L$ so that

- s_0 approximates f
- s_i is close to s_{i+1}
- s_{k-1} approximates f

By Lemma 17.12 this implies $f_* = (f')_*$ as required.

The space $|L|$ has an open cover

$$\mathcal{V} = \{\mathrm{star}_L^\circ(v) \mid v \text{ is a vertex of } L\}$$

and since $|L|$ is compact we may choose a Lebesgue number $\delta > 0$ for this cover. On the other hand, by compactness of $|K|$ there is a k so that

$$d(H_{i/k}(x), H_{(i+1)/k}(x)) < \delta \text{ for all } i \text{ and all } x \in |K|.$$

We show that there is then a simplicial approximation $s_i : K^{(m)} \longrightarrow L$ of both $p = H_{i/k}$ and $q = H_{(i+1)/k}$. Indeed, it suffices to choose an m for each i

separately and then take the maximum. Note that this guarantees s_i and s_{i+1} are close by Lemma 17.11 since they are both approximations of $q = H_{(i+1)/k}$.

The key is to note that

$$\{p^{-1}(\text{star}^\circ_L(v)) \cap q^{-1}(\text{star}^\circ_L(v) \mid v \in V(L)\}$$

is an open cover of $|K|$. For this we need only show it is a cover: that any $x \in |K|$ lies in one of the sets. By choice of k, we have $d(p(x), q(x)) < \delta$, and since δ is a Lebesgue number $\{p(x), q(x)\} \subseteq \text{star}^\circ_L(v)$ for some v. Hence $x \in p^{-1}(\text{star}^\circ_L(v)) \cap q^{-1}(\text{star}^\circ_L(v))$ as required.

Now to define the common simplicial approximation $s_i : K^{(m)} \longrightarrow L$ to p and q we proceed as in the Simplicial Approximation Theorem (16.8). We subdivide K until $\text{mesh}(K^{(m)}) < \delta/2$ and then $\text{diam}(\text{star}^\circ_{K^{(m)}}(u)) < \delta$ for all vertices u.

Thus, for all vertices u there is a vertex v of L so that

$$\text{star}^\circ_{K^(m)}(u) \subseteq p^{-1}(\text{star}^\circ_L(v)) \cap q^{-1}(\text{star}^\circ_L(v)).$$

Now define $s_i(u) = v$, and by the same argument as in the Simplicial Approximation Theorem 16.8 it is a simplicial map.