14 The Mayer-Vietoris sequence

We have seen that it is routine but extremely laborious to calculate the homology of complexes from first principles.

Eventually we will prove that $H_*(K)$ only depends on the geometric realizations and we would much prefer not to have to work with specific triangulation. In other words we'd like to think of a space as built up from more complicated blocks than simplices. This might well mean that the building blocks themselves have some homology.

In other words, if $X = A \cup B$ we would like to calculate $H_*(X)$ in terms of $H_*(A), H_*(B)$ and $H_*(A \cap B)$. This is what the Mayer-Vietoris Theorem does. As usual with homology, we will prove a little algebraic result and then deduce the statement about spaces from it.

14.1 Exact sequences of chain complexes

The algebraic input is a short exact sequence of chain complexes.

Definition 14.1. *A* short exact sequence *of chain complexes is a sequence*

$$0 \longrightarrow A_\bullet \xrightarrow{\theta} B_\bullet \xrightarrow{\phi} C_\bullet \longrightarrow 0$$

of chain complexes and chain maps so that for each integer n, we have a short exact sequence

$$0 \longrightarrow A_n \xrightarrow{\theta_n} B_n \xrightarrow{\phi_n} C_n \longrightarrow 0$$

of abelian groups.

Of course we always have one boring example.

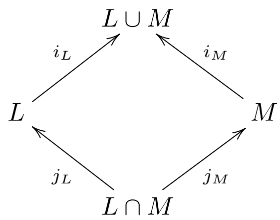
Example 14.2. *The trivial short exact sequence of chain complexes is*

$$0 \longrightarrow A_\bullet \xrightarrow{\theta} A_\bullet \oplus C_\bullet \xrightarrow{\phi} C_\bullet \longrightarrow 0$$

where θ includes $A_\bullet$ as the first factor and ϕ is projection onto the second factor.

The example of most immediate interest for us is as follows.

Example 14.3. *Suppose K is a simplicial complex which can be written as a union of two other simplicial chain compexes: $K = L \cup M$. We introduce* *notation for the inclusion maps:*



There is a short exact sequence of chain complexes

$$0 \longrightarrow C_\bullet(L \cap M) \xrightarrow{\{(j_L)_\sharp, -(j_M)_\sharp\}} C_\bullet(L) \oplus C_\bullet(M) \xrightarrow{\langle (i_L)_\sharp, (i_M)_\sharp \rangle} C_\bullet(L \cup M) \longrightarrow 0.$$

Indeed, we know that the maps $(i_L)_\sharp, (i_M)_\sharp, (j_L)_\sharp$ *and* $(j_M)_\sharp$ *are chain maps, so it remains only to note that since* $K_n = L_n \cup M_n$ *is the disjoint union of* L_n *and* M_n *with the interesection identified, considering basis elements, we have a short exact sequence*

$$0 \longrightarrow \mathbb{Z}[L_n \cap M_n] \xrightarrow{\{(j_L)_\sharp, -(j_M)_\sharp\}} \mathbb{Z}[L_n] \oplus \mathbb{Z}[M_n] \xrightarrow{\langle (i_L)_\sharp, (i_M)_\sharp \rangle} \mathbb{Z}[L_n \cup M_n] \longrightarrow 0.$$

14.2 Statement of the Mayer-Vietoris theorem

Finally we are ready to state the theorem.

Theorem 14.4. *(Mayer-Vietoris) A short exact sequence*

$$0 \longrightarrow A_\bullet \xrightarrow{\theta} B_\bullet \xrightarrow{\phi} C_\bullet \longrightarrow 0$$

of chain complexes induces a long exact sequence of homology. In more detail, there are natural connecting homomorphisms

$$\partial : H_n(C_\bullet) \longrightarrow H_{n-1}(A_\bullet)$$

so that the sequence

$$H_n(A) \xrightarrow{\theta_*} H_n(B) \xrightarrow{\phi_*} H_n(C) \xrightarrow{\partial}$$

$$H_{n-1}(A) \xrightarrow{\theta_*} H_{n-1}(B) \xrightarrow{\phi_*} H_{n-1}(C) \xrightarrow{\partial}$$

is exact.

In fact the statement has innumerable algebraic applications, but for us it will be enough to consider one topological application.

Corollary 14.5. *If $K = L \cup M$ there is a long exact sequence*

$$H_n(L\cap M) \xrightarrow{\{(j_L)_\sharp, -(j_M)_\sharp\}} H_n(L)\oplus H_n(M) \xrightarrow{\langle (i_L)_\sharp, (i_M)_\sharp\rangle} H_n(L\cup M) \xrightarrow{\partial}$$

$$H_{n-1}(L\cap M) \xrightarrow{\{(j_L)_\sharp, -(j_M)_\sharp\}} H_{n-1}(L)\oplus H_{n-1}(M) \xrightarrow{\langle (i_L)_\sharp, (i_M)_\sharp\rangle} H_{n-1}(L\cup M) \xrightarrow{\partial}$$

14.3 Examples of the Mayer-Vietoris theorem

Before proving the theorem we will give a range of examples. The reader is encouraged to give many more.

The general method is as follows

Step 1: Express the space as a union $X = A \cup B$ where one understands A, B and $A \cap B$.

Step 2: Work out the homotopy types of A, B and $A\cap B$, and hence also write down their homologies.

Step 3: Write out the entire Mayer-Vietoris sequence.

Step 4: Work out what happens in H_0 using knowledge of the path components.

Step 5: Extract from the Mayer-Vietoris sequence exact sequencess for the homology groups $H_n(A \cup B)$.

Step 5: Identify the maps $(j_L)_*$ and $(j_M)_*$.

Step 6: Conclude as much as possible about $H_*(A\cup B)$ from the exact sequences.

Example 14.6. *The homology of the projective plane is given by*

$$H_i(\mathbb{R}P^2) = \begin{cases} 0 & i=2 \\ \mathbb{Z}/2 & i=1 \\ \mathbb{Z} & i=0 \\ 0 & \textit{otherwise} \end{cases}$$

Proof. We view the projective plane as the union of a Möbius strip A and a closed disc B. We could give an explicit triangulation, but this would be a distraction.

It is clear that $B \simeq *$ and that $A \cap B =: S^1_\partial$ is a circle. It is also easy to see that the Möbius strip A is homotopy equivalent to its central circle S^1_A.

Thus A, B and $A \cap B$ are all path connected (with H_0 being $\mathbb{Z}$) and all have $H_i = 0$ for $i \geq 2$. Finally $H_1(B) \cong H_1(pt) = 0$,

$$H_1(A \cap B) = H_1(S^1_\partial) \cong \mathbb{Z} \text{ and } H_1(A) \cong H_1(S^1_A) \cong \mathbb{Z}.$$

We may now write out the Mayer-Vietoris sequence and find that there is an exact sequence

$$0 \longrightarrow H_2(\mathbb{R}P^2) \longrightarrow H_1(S^1_\partial) \xrightarrow{\alpha} H_1(S^1_A) \longrightarrow H_1(\mathbb{R}P^2) \longrightarrow 0.$$

Finally $H_1(S^1_\partial)$ is generated by the cycle of edges, which includes as the boundary circle of the Möbius strip. This then retracts to the cycle going twice round the central cirle S^1_A. and α is multiplication by 2, giving the claimed conclusion. □

Before proceeding it is worth recording another general result.

Lemma 14.7. *(a) If $K = L \coprod M$ is the disjoint union of subcomplexes L and M then*

$$H_i(K) \cong H_i(L) \oplus H_i(M)$$

for all i.

(b) If $K = L \vee M$ is the wedge (one point union) of L and M then

$$H_i(K) \cong H_i(L) \oplus H_i(M)$$

for all $i \geq 1$. Of course $H_0(L \vee M)$ is easy to work out (it is of rank one less than $H_0(L) \oplus H_0(M)$).

Proof. For Part (a) we apply the Mayer-Vietoris sequence, noting that $L \cap M = \emptyset$ so that $H_i(L \cap M) = 0$ for all i. Accordingly the map

$$H_i(L) \oplus H_i(M) \longrightarrow H_i(L \coprod M)$$

is an isomorphism. [Of course it is also obvious from the definition!].

For Part (b) the argument is almost identical except that $L \cap M = pt$ has homology in degree 0. □

Example 14.8. *The homology of the compact orientable surface of genus $g \geq 0$ is given by*

$$H_i(M(g)) = \begin{cases} \mathbb{Z} & i = 2 \\ \mathbb{Z}^{2g} & i = 1 \\ \mathbb{Z} & i = 0 \\ 0 & \textit{otherwise} \end{cases}$$

Proof. The case $g = 0$ is the sphere, which has been dealt with above. For $g \geq 1$, we view $M(g)$ as formed from a $4g$-gon with edges stuck together according to the surface word

$$[x_1, y_1][x_2, y_2] \cdots [x_g, y_g], \text{ where } [a, b] = aba^{-1}b^{-1}.$$

We decompose this into a small central disc B and the complement A of its interior. We could give an explicit triangulation, but this would be a distraction.

It is clear that $B \simeq *$ and that $A \cap B =: S^1_\partial$ is a circle. It is also easy to see that the boundary $4g$-gon is a strong deformation retract of A, and (after sticking the edges together) this is a wedge of $2g$ circles, which are labelled by the $2g$ letters $x_1, y_1, x_2, y_2, \ldots, x_g, y_g$.

Thus A, B and $A \cap B$ are all path connected (with H_0 being $\mathbb{Z}$) and all have $H_i = 0$ for $i \geq 2$. Finally $H_1(B) \cong H_1(pt) = 0$,

$$H_1(A \cap B) = H_1(S^1_\partial) \cong \mathbb{Z} \text{ and } H_1(A) \cong H_1(\bigvee_{i=1}^{2g} S^1) \cong \mathbb{Z}^{2g},$$

where the last statement used Lemma **??**.

We may now write out the Mayer-Vietoris sequence and find that there is an exact sequence

$$0 \longrightarrow H_2(M(g)) \longrightarrow H_1(S^1_\partial) \xrightarrow{\alpha} H_1(\bigvee_{i=1}^{2g} S^1) \longrightarrow H_1(M(g)) \longrightarrow 0.$$

Finally $H_1(S^1_\partial)$ is generated by the cycle of edges, which retracts along the $4g$-gon, where it maps to

$$(x_1 + y_1 - x_1 - y_1) + (x_2 + y_2 - x_2 - y_2) + \cdots + (x_g + y_g - x_g - y_g) = 0$$

(note this is the abelianized version of the surface word). This gives $\alpha = 0$, and hence the claimed conclusion. □

The point of the next example is to show that different geometric decompositions of a space (in this case the torus) can give quite different calculations, and also to show that the knowledge of $H_*(A), H_*(B)$ and $H_*(A \cap B)$ does not completely determine $H_*(A \cup B)$. Indeed the torus and Klein bottle have different homologies, but are both decomposed into subspaces which are homeomorphic.

Example 14.9. *The homology of the torus is given by*

$$H_i(T^2) = \begin{cases} \mathbb{Z} & i = 2 \\ \mathbb{Z}^2 & i = 1 \\ \mathbb{Z} & i = 0 \\ 0 & \textit{otherwise} \end{cases}$$

The homology of the Klein bottle is given by

$$H_i(K^2) = \begin{cases} 0 & i = 2 \\ \mathbb{Z} \oplus \mathbb{Z}/2 & i = 1 \\ \mathbb{Z} & i = 0 \\ 0 & \textit{otherwise} \end{cases}$$

Proof. Let X denote either T^2 or K^2. In either case we decompose X into two cylinders A and B, which intersect each other in two circles: $A \cap B = S^1_x \coprod S^1_y$. We could give an explicit triangulation, but this would be a distraction.

Since A and B are cylinders, they are each homotopy equivalent to a circle, for example that at one end: $A \simeq S^1_A$ and $B \simeq S^1_B$. It is visible that that $A \cap B =: S^1_x \coprod S^1_y$.

Thus A, B are path connected (with H_0 being $\mathbb{Z}$) and $A \cap B$ has two path components and $H_0 \cong \mathbb{Z}^2$. It is also clear that all have $H_i = 0$ for $i \geq 2$. Finally $H_1(A) \cong \mathbb{Z}$, $H_1(B) \cong \mathbb{Z}$, and $H_1(A \cap B) = H_1(S^1_x \coprod S^1_y) \cong \mathbb{Z}^2$.

We may now write out the Mayer-Vietoris sequence and find that there is an exact sequence

$$0 \longrightarrow H_2(M(g)) \longrightarrow H_1(S^1_x \coprod S^1_y) \xrightarrow{\alpha} H_1(S^1_A) \oplus H_1(S^1_B) \longrightarrow H_1(X) \longrightarrow \mathbb{Z} \longrightarrow 0.$$

It follows that

$$H_2(X) = \ker(\alpha)$$

and

$$H_1(X) = \mathbb{Z} \oplus [\mathbb{Z}^2 / \operatorname{Im}(\alpha)].$$

It remains to identify α, and this is a matter of carefully identifying generators. Since all four inclusions

$$S^1_x \longrightarrow A \simeq S^1_A, S^1_y \longrightarrow A \simeq S^1_A, S^1_x \longrightarrow B \simeq S^1_B \text{ and } S^1_y \longrightarrow B \simeq S^1_B$$

are all homotopy equivalences, there is really only one choice we need to make, but then we need to think carefully about its implications.

First we choose a representative cycle z_x for a generator $[z_x] \in H_1(S^1_x)$. We use this to give us a generators

- $z_A = (j_A)_\sharp(z_x)$ of $H_1(S^1_A)$
- $z_B = (j_B)_\sharp(z_x)$ of $H_1(S^1_B)$ and
- z_y for $H_1(S^1_y)$ chosen so that $(j_A)_*([z_y]) = [z_A]$

We now have generators for all our groups, and we have

$$\alpha = \begin{pmatrix} 1 & 1 \\ -1 & ? \end{pmatrix}$$

The remaining question is how $(j_B)_*[z_y]$ compares to our generator $[z_B]$ of $H_1(B)$.

Observation

- For the torus $(j_B)_*([z_y]) = +[z_B]$ in $H_1(S^1_B)$
- For the Klein bottle $(j_B)_*([z_y]) = -[z_B]$ in $H_1(S^1_B)$

Accordingly

$$\alpha_T = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \text{ and } \alpha_K = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

The result follows for the torus since α_T has $(1, -1) = [z_x] - [z_y]$ generating the kernel, and the image (generated by $(1, -1) = ([z_A], -[z_B])$) is a direct summand.

The result follows for the Klein bottle since α_K has determinant 2, so it is injective, and the image is of index 2. □

Example 14.10. *For $n \geq 0$, the homology of complex projective n-space is given by*

$$H_i(\mathbb{C}P^n) = \begin{cases} \mathbb{Z} & \textit{if } i \textit{ is even and } 0 \leq i \leq 2n \\ 0 & \textit{otherwise} \end{cases}$$

Proof. We may prove this by induction on n. Indeed, if $n = 0$ the space $\mathbb{C}P^0$ is a point, and the statement is correct. Now suppose $n \geq 1$ and that $H_*(\mathbb{C}P^{n-1})$ is as stated.

We take B to be a small closed $2n$-disc around $[0:0:\cdots:0:1]$ and A to be the complement of the interior. We see that A consists of points $[z_0 : z_1 : \cdots : z_{n-1} : z_n]$ with not all of $z_0, z_1, \ldots, z_{n-1}$ are 0, so that it is homotopy equivalent to

$$\mathbb{C}P^{n-1} =$$

$\{\ [\mathrm{z}_0 : z_1 : \cdots : z_{n-1} : 0] \mid$ not all entries are zero $\}$.

Thus $A \simeq \mathbb{C}P^{n-1}$, $B \simeq *$ and $A \cap B = S^{2n-1}$.

As usual, H_0 is immediate, and since $H_i(S^{2n-1}) = 0$ for $0 < i < 2n-1$, the Mayer-Vietoris sequence gives

$$H_i(\mathbb{C}P^n) \cong H_i(\mathbb{C}P^{n-1}) \text{ for } i \leq 2n-1$$

and

$$H_{2n}(\mathbb{C}P^n) = \mathbb{Z}.$$

This completes the inductive step and the result follows by the principle of mathematical induction. □

14.4 Proof of the Mayer-Vietoris theorem

We need to define the boundary map and then check exactness in 3 places. To start with, we display three rows of the chain complexes to work on

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_{n+1} & \xrightarrow{\theta_{n+1}} & B_{n+1} & \xrightarrow{\phi_{n+1}} & C_{n+1} & \longrightarrow & 0 \\
 & & \downarrow d & & \downarrow d & & \downarrow d & & \\
0 & \longrightarrow & A_n & \xrightarrow{\theta_n} & B_n & \xrightarrow{\phi_n} & C_n & \longrightarrow & 0 \\
 & & \downarrow d & & \downarrow d & & \downarrow d & & \\
0 & \longrightarrow & A_{n-1} & \xrightarrow{\theta_{n-1}} & B_{n-1} & \xrightarrow{\phi_{n-1}} & C_{n-1} & \longrightarrow & 0
\end{array}$$

14.4.1 Definition of ∂

The first thing is to define $\partial : H_n(C) \longrightarrow H_{n-1}(A)$. Crudely speaking we take

$$\partial(\gamma) =! = \theta^{-1} \circ d \circ \phi^{-1}(\gamma)$$

but we need to make sense of this. More precisely, we choose a representative cycle $z \in C_n$ for γ, so that $\gamma = [z]$ and then choose $y \in B_n$ so that $\phi y = z$. Now we note that since z is a cycle, $\phi(dy) = d\phi y = dz = 0$ and so there is an $x \in A_{n-1}$ so that $\theta(x) = dy$. Finally, we argue that x is a cycle: indeed $\phi dx = d\phi x = ddy = 0$ so $dx = 0$ since ϕ is injective. In short we may display this

$$\begin{array}{ccccc} & & y & \longmapsto & z \\ & & \downarrow & & \downarrow \\ x & \longmapsto & dy & \longmapsto & 0 \\ \downarrow & & \downarrow & & \\ dx & \longmapsto & 0 & & \end{array}$$

Next we need to check that this is well defined, since we made choices of z and of y.

The first choice was that of cycle representative z. If we had chosen z' instead then there would be a $c \in C_{n+1}$ so that $d(c) = z - z'$, and then a $b \in B_{n+1}$ so that $\phi(b) = c$. Then we could choose $y' = y - db$ as our lift of z' and then $dy' = dy$, so that we would get the same element x.

Our second choice was of y. If we had instead chosen y' with $\phi y' = z$ then of course there is an a with $\theta a = y - y'$, and so $da = x - x'$ and so x and x' define the same homology class.

Altogether we have shown that ∂ is a well defined group homomorphism not depending on any choices.

14.4.2 Exactness at A

We need to check that $\operatorname{Im} \partial = \ker \theta_*$ at $H_{n-1}(A)$. First note that $\theta_* \circ \partial = 0$ since with the above notation $\theta_*[x] = [\theta x] = [dy] = [0]$. Next suppose that $\alpha \in H_{n-1}(A)$ has $\theta_*(\alpha) = 0$. Choose a cycle representative so that $\alpha = [x]$. We have $\theta_*[x] = 0$, so that $\theta(x) = dy$ for some $y \in B_n$. Then if we take $z = \phi y$ we get $\partial[z] = [x] = \alpha$ as required.

14.4.3 Exactness at B

We need to check that $\operatorname{Im} \theta_* = \ker \phi_*$ at $H_n(B)$. First note that $\phi_* \circ \theta_* = 0$ since $\phi \circ \theta = 0$. Now if $\phi_* \beta = 0$ we may choose a representative cycle $y \in B_n$

with $\beta = [y]$, and there is an $c \in C_{n+1}$ with $dc = \phi(y)$.

$$x \longmapsto y - db, \qquad \begin{array}{ccc} b & \longmapsto & c \\ & & \big\downarrow \\ y & \longmapsto & \phi y \\ \big\downarrow & & \\ 0 & & \end{array}$$

Since ϕ is surjective there is a b so that $\phi(b) = c$. Then $\phi(y - db) = 0$ so that there is an $x \in A_n$ with $\theta(x) = y - db$. Furthermore, $\theta(dx) = d\theta x = dy - d^2 b = 0$ so that since θ is injective $dx = 0$ and x is a cycle, so $\theta_*[x] = [\theta x] = [y - db] = [y]$ as required.

14.4.4 Exactness at C

We need to check that $\operatorname{Im} \phi_* = \ker \partial$ at $H_n(C)$. First note that $\partial \circ \phi_* = 0$ since if $\beta \in H_n(B)$ has representative cycle $y \in B_n$ we have $\phi_*(\beta) = \phi_*[y] = [\phi y]$ and in defining $\partial([z])$ with $z = \phi y$ we get $dy = 0$ so $x = 0$.

Finally if $\partial[z] = [x] = 0$ we have $x = da$ for some $a \in A_n$.

$$\begin{array}{ccccc} a & & y & \longmapsto & z \\ \big\downarrow & & \big\downarrow & & \\ x & \longmapsto & dy & & \end{array}$$

We now modify our choice of y by taking $y' = y - \theta a$ (noting this still maps to z). We then check y' is a cycle since $d(y - \theta a) = dy - d\theta a = dy - \theta da = x - x = 0$. Since y' is a cycle we have $\phi_*[y'] = [z]$ as required.

15 The Euler characteristic and the Lefschetz fixed point theorem

We give two examples where a simplicial invariant can be reduced to homology. In view of the fact that simplicial homology is a homotopy invariant, this is extremely powerful.

The Euler characteristic is a rather well known classical invariant (going back to Euler's formula $V - E + F = 2$ for polyhedra). From the point of view of combinatorics, the amazing thing is that this combination of numbers is invariant. From the point of view of topology the amazing thing is that a topological invariant can be calculated in such concrete terms. Probably the first application is the fact that orientable surfaces are distinguished by their Euler characteristice ($\chi(M(g)) = 2 - 2g$).

The Lefschetz Fixed Point Theorem builds on this. In fact it gives a sufficient condition for a self-map $f : X \longrightarrow X$ of a triangulable space X to have a fixed point. It is amazingly powerful, and it also has counterparts in other parts of mathematics (e.g., arithmetic geometry where proofs of the Weil Conjectures involve this type of construction).

15.1 Homology with rational coefficients

We briefly note that we may systematically throw away all the torsion.

Definition 15.1. *If K is a simplicial complex and F is a field, we define*

$$C_n(K; F) = F[K_n]$$

(i.e., exactly the same as for $C_n(K)$ but with $\mathbb{Z}$ replaced by F). We make this into a chain complex $C_\bullet(K; F)$ using exactly the same formula for d_n as for the integral case. We then define

$$H_n(K; F) = H_n(C_\bullet(K; F)).$$

We note that this actually makes sense for any ring F, but we will concentrate on fields, and on $F = \mathbb{Q}$ in particular.

Lemma 15.2. *If $H_n(K) = \mathbb{Z}^r \oplus T$ with T a finite abelian group then $H_n(K; \mathbb{Q}) = \mathbb{Q}^r$.*

Remark 15.3. Of course this means that $H_*(K; \mathbb{Q})$ determines the torsion free part of $H_*(K)$. One can use this along with $H_*(K; \mathbb{F}_p)$ to give information about the p-torsion of $H_*(K)$.

Proof. The key is that $B_i(K)$ and $Z_i(K)$ are free abelian groups, and if we choose bases for them, they also give bases for $B_i(K; \mathbb{Q})$ and $Z_i(K; \mathbb{Q})$.

If $H_n(K) = \mathbb{Z}^r \oplus T$ this means that the rank of $Z_n(K)$ is r more than the rank of $B_n(K)$. Here the sentence was written with 'rank' meaning 'the number of generators in a $\mathbb{Z}$-basis'. This means that the dimension of $Z_n(K; \mathbb{Q})$ is r more than the dimension of $B_n(K)$. □

15.2 The Euler characteristic

We now note that we may define two versions of the Euler characteristic.

Definition 15.4. *To a simplicial complex K we may associate two numbers.*

(i) The combinatorial Euler characteristic *is defined by*

$$\chi_{comb}(K) = \sum_i (-1)^i |K_i|,$$

where $|K_i|$ is the number of i-simplices.

(i) The homological Euler characteristic *is defined by*

$$\chi_{hom}(K) = \sum_i (-1)^i \dim_{\mathbb{Q}}(H_i(K;\mathbb{Q})).$$

In a classic sort of theorem, we find that the number defined in terms of an invariant (the homology) can be calculated using a choice of additional structure (the triangulation).

Theorem 15.5. *For any simplicial complex K we find*

$$\chi_{comb}(K) = \chi_{hom}(K).$$

Remark 15.6. Once we have shown that homology is a homotopy type invariant, this shows that $\chi_{comb}(K)$ is a homotopy type invariant, and in particular does not depend on triangulation.

Proof. First note that

$$B_i(K;\mathbb{Q}) \subseteq Z_i(K;\mathbb{Q}) \subseteq C_i(K;\mathbb{Q}).$$

Choose a basis $\{b_i^\alpha\}_{\alpha=1}^{m_i}$ of $B_i(K;\mathbb{Q})$ (with m_i elements, say) extend this to a basis of $Z_i(K;\mathbb{Q})$ by adding $\{z_i^\beta\}_{\beta=1}^{n_i}$ (with n_i elements, say) extend this to a basis of $C_i(K;\mathbb{Q})$ by adding $\{c_i^\gamma\}_{\gamma=1}^{p_i}$ (with p_i elements, say).

Indeed we can refine this a little by starting with the highest dimension of cells, and then when we reach the i-simplices (having already made choices in higher dimensions) we note that $m_i = p_{i+1}$ and we may take $b_i^\gamma = d(c_{i+1}^\gamma)$.

Now note

$$\begin{aligned}\chi_{comb}(K) &= \sum_i (-1)^i |K_i| \\ &= \sum_i (-1)^i (m_i + n_i + p_i) \\ &= \sum_i (-1)^i (m_i - p_{i+1}) + \sum_i (-1)^i n_i \\ &= \sum_i (-1)^i n_i \\ &= \sum_i (-1)^i \dim_{\mathbb{Q}} H_i(K;\mathbb{Q}) \\ &= \chi_{hom}(K)\end{aligned}$$

□

In view of the theorem we will usually write $\chi(X)$ for the Euler characteristic and freely use the fact that we may calculate it with combinatorics or homology.

Example 15.7. *The Euler characteristics of the compact connected surfaces are*

$$\chi(M(g)) = 2 - 2g$$

$$\chi(N(h)) = 2 - h.$$

Exercise 15.8. *Count the number of edges, vertices and faces for the 5 platonic solids, and hence $\chi = V - E + F$. Choose a triangulation of the 2-torus and hence calculate its Euler characteristic.*

15.3 The Lefschetz Fixed Point Theorem

We now want to apply similar ideas to study maps.

Definition 15.9. *If $A_\bullet$ is a chain complex of rational vector spaces and $\theta : A_\bullet \longrightarrow A_\bullet$ then its* Lefschetz number *is defined by*

$$\Lambda(\theta) = \sum_i (-1)^i \mathrm{tr}(\theta_i : A_i \longrightarrow A_i).$$

Using the topological invariance of homology we have the following wonderful theorem.

Theorem 15.10. (Lefschetz Fixed Point Theorem (LFPT)) *Suppose X is a triangulable space and $f : X \longrightarrow X$ is a continuous self-map inducing $f_* : H_*(X) \longrightarrow H_*(X)$. If $\Lambda(f_*) \neq 0$ then f has a fixed point.*

For the present we will prove the statement for simplicial map, returning later to deduce the full LFPT when we have considered simplicial approximations.

Theorem 15.11. (Simplicial Lefschetz Fixed Point Theorem (SLFPT)) *Suppose K is a simplicial complex and $s : K \longrightarrow K$ is a simplicial self-map inducing $s_* : H_*(K) \longrightarrow H_*(K)$. If $\Lambda(s_*) \neq 0$ then $|s| : |K| \longrightarrow |K|$ has a fixed point.*

Indeed, we note that if s takes any simplex $\langle v_0, \ldots, v_k \rangle$ to itself then it just permutes the vertices so $|s|$ fixes the barycentre $\hat{\sigma}$. Thus if $|s|$ has no fixed point then every map

$$s_\sharp : C_i(K) \longrightarrow C_i(K)$$

has zero trace and hence $\Lambda(s_\sharp) = 0$. The result follows from an algebraic proposition.

Proposition 15.12. *If $s : K \longrightarrow K$ is a simplicial map inducing $s_\sharp : C_\bullet(K;\mathbb{Q}) \longrightarrow C_\bullet(K;\mathbb{Q})$ and $s_* : H_*(K;\mathbb{Q}) \longrightarrow H_*(K;\mathbb{Q})$ then*

$$\Lambda(s_\sharp) = \Lambda(s_*).$$

Proof. As in the proof of 15.5 that the combinatorial and homological Euler characteristics agree, we note that

$$B_i(K;\mathbb{Q}) \subseteq Z_i(K;\mathbb{Q}) \subseteq C_i(K;\mathbb{Q}),$$

and we choose a basis $\{b_i^\alpha\}_{\alpha=1}^{m_i}$ of $B_i(K;\mathbb{Q})$ (with m_i elements, say) extend this to a basis of $Z_i(K;\mathbb{Q})$ by adding $\{z_i^\beta\}_{\beta=1}^{n_i}$ (with n_i elements, say) extend this to a basis of $C_i(K;\mathbb{Q})$ by adding $\{c_i^\gamma\}_{\gamma=1}^{p_i}$ (with p_i elements, say). Once again we start from the top, note that $m_i = p_{i+1}$ and we take $b_i^\gamma = d(c_{i+1}^\gamma)$.

Now consider the matrix of $s_\sharp$ with respect to these bases

$$\Theta_i = \left(\begin{array}{c|c|c} R_i & ? & ? \\ \hline ? & S_i & ? \\ \hline ? & ? & T_i \end{array} \right)$$

Now from our choice of bases and the fact that θ is a chain map we see

$$T_i = R_{i-1}$$

Now we calculate

$$\begin{aligned}
\Lambda(s_\sharp) &= \textstyle\sum_i (-1)^i \mathrm{tr}(s_\sharp : C_i(K;\mathbb{Q}) \longrightarrow C_i(K;\mathbb{Q})) \\
&= \textstyle\sum_i (-1)^i \mathrm{tr}(\Theta_i) \\
&= \textstyle\sum_i (-1)^i (\mathrm{tr}(R_i) + \mathrm{tr}(S_i) + \mathrm{tr}(T_i)) \\
&= \textstyle\sum_i (-1)^i (\mathrm{tr}(R_i) - \mathrm{tr}(T_{i+1})) + \sum_i (-1)^i \mathrm{tr}(S_i) \\
&= \textstyle\sum_i (-1)^i \mathrm{tr}(s_* : H_i(K;\mathbb{Q}) \longrightarrow H_i(K;\mathbb{Q})) \\
&= \Lambda(s_*)
\end{aligned}$$

□

15.4 Applications of the Lefschetz Fixed Point Theorem

Corollary 15.13. *If X is a triangulable space with a self-map $f : X \longrightarrow X$ with no fixed points and $f \simeq id_X$ then $\chi(X) = 0$.*

Proof. We have

$$0 = \Lambda(f) = \Lambda(id) = \chi(X),$$

where the first equality is from the LFPT since f has no fixed points, the second follows since $f \simeq id$ so that $f_* = id$, and the third follows since the trace of the identity is the dimension. □

Corollary 15.14. *If X has a free action of the circle group then $\chi(X) = 0$.*

Proof. The map $m_\theta : X \longrightarrow X$ given by the action of $e^{i\theta}$ gives a self-map. Since the action is free, m_θ has no fixed point if θ is not a multiple of 2π. The maps $m_{t\theta}$ for $t \in [0, 1]$ give a homotopy $m_\theta \simeq m_0 = id_X$. □

Example 15.15. *The torus and Klein bottle have fixed point free self-maps homotopic to the identity. No other surfaces do (since they have non-zero Euler characteristic).*

Example 15.16. *Odd spheres have have fixed point free self-maps homotopic to the identity. Even spheres do not.*

The antipodal map $f : S^n \longrightarrow S^n$ is of degree $(-1)^{n+1}$