12 Homology of simplicial complexes

We are now ready to define the homology of simplicial complexes. Unless we say otherwise, all our simplicial complexes K will be abstract simplicial complexes, but we will tend to draw pictures of geometric simplicial complexes $\overline{K}$ whose associated abstract complexes are K.

The strategy for defining homology is as follows.

Step 1: Start with a simplicial complex K.

Step 2: Construct a chain complex $C_\bullet(K)$.

Step 3: Take the homology of the chain complex: $H_*(K) = H_*(C_\bullet(K))$.

It will be easy to see that

1. if $s : K \longrightarrow L$ is a simplicial map then
2. s induces a chain map $s_\bullet : C_\bullet(K) \longrightarrow C_\bullet(L)$ and then
3. $s_\bullet$ induces a map $s_* : H_*(K) \longrightarrow H_*(L)$.

Eventually we will show

- $H_*(K)$ depends only on the topological space $|K|$.
- Any continuous function $f : |K| \longrightarrow |L|$ (unrelated to the simplical structure!) induces a group homomorphism $f_* : H_*(|K|) \longrightarrow H_*(|L|)$ in a functorial way.
- If $f \simeq f'$ then $f_* = f'_*$

12.1 The chain complex

If K is a simplical complex, we define $C_n(K) := \mathbb{Z}[K_n]$ (i.e., the free abelian group with basis given by the n-simplices of K). Note that this means that $C_i(K) = 0$ if $i < 0$ or if $i > \dim(K)$.

For each $n \geq 0$ we now define a map

$$d_n : C_n(K) \longrightarrow C_{n-1}(K).$$

To do this, we *choose* an ordering of the vertices. We will shortly show this makes no difference. If $\{v_0, v_1, \ldots, v_n\}$ is an n-simplex with the vertiices in the right order we write $\langle v_0, v_1, \dots, v_n\rangle$ for the corresponding basis element of $C_n(K) = \mathbb{Z}[K_n]$.

Definition 12.1. *The map $d_n : C_n(K) \longrightarrow C_{n-1}(K)$ is defined by*

$$\begin{aligned} d_n(\langle v_0, v_1, \dots, v_n\rangle) &= \\ \langle v_1, \dots, v_n\rangle - \langle v_0, v_2, \dots, v_n\rangle &+ \cdots + (-1)^n d_n(\langle v_0, v_1, \dots, v_{n-1}\rangle \\ &= \sum_{i=0}^{n} (-1)^i \langle v_1, \dots, \hat{v}_i, \dots, v_n\rangle \end{aligned}$$

where $\hat{v}_i$ means that v_i is omitted.

Remark 12.2. (i) Since $C_n(K)$ is a free abelian group, any homomorphism (such as d_n) with that domain is freely determined by where the basis elements go.

(ii) Note that if we permit other orderings of the simplex (specified by a permutation σ of $\{0, \dots, n\}$, and relate them to the chosen basis by

$$\langle v_{\sigma(0)}, v_{\sigma(1)}, \dots v_{\sigma(n)}\rangle = \mathrm{sign}(\sigma)\langle v_0, v_1, \dots v_n\rangle$$

then the formula is exactly the same. This shows that not only does the answer not depend on the total order of the vertices, it does not depend on the ordering of the vertices within a simplex.

Lemma 12.3. *For any n and any simplical complex K, $d_{n-1} \circ d_n = 0$.*

Proof. It suffices to verify that each of the basis elements get sent to zero. We simply calculate

$$\begin{aligned} d_{n-1}(d_n\langle v_0, v_1, \dots, v_n\rangle) &= d_{n-1}(\textstyle\sum_{i=0}^n (-1)^i \langle v_0, v_1, \dots \hat{v}_i, \dots, v_n\rangle) \\ &= \textstyle\sum_{i=0}^n (-1)^i d_{n-1}(\langle v_0, v_1, \dots \hat{v}_i, \dots, v_n\rangle) \\ &= \textstyle\sum_{i=0}^n (-1)^i \sum_{j=0}^{i-1} (-1)^j \langle v_0, v_1, \dots, \hat{v}_j, \dots, \hat{v}_i, \dots, v_n\rangle) \\ &\quad + \textstyle\sum_{i=0}^n (-1)^i \sum_{j=i+1}^{n} (-1)^{j-1} \langle v_0, v_1, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_n\rangle \end{aligned}$$

Now notice that if we pick $0 \le s < t \le n$ then the simplex $\langle v_0, v_1, \dots, \hat{v}_s, \dots, \hat{v}_t, \dots, v_n\rangle$ occcurs twice:

once with $i = s, j = t$ and sign $(-1)^{s+t-1}$

once with $j = s, i = t$ and sign $(-1)^{s+t}$

These two terms cancel. ∎

12.2 Homology of simplicial complexes

In the light of Lemma 12.3 we may make the following definition.

Definition 12.4. *If K is an abstract simplicial complex, the homology is defined by*

$$H_n(K) := H_n(C_\bullet(K), d)$$

Explicitly

$$H_n(K) := \frac{\ker(d : C_n(K) \longrightarrow C_{n-1}(K))}{\operatorname{Im}(d : C_{n+1}(K) \longrightarrow C_n(K))}.$$

We note first that it is obvious that $H_i(K) = 0$ if $i < 0$ or if $i > \dim(K)$. Let us look at a few more elementary examples.

Example 12.5. *(a) The first example we have already done $K = \partial\Delta^2$:*

$$H_n(\partial\Delta^2) = \begin{cases} \mathbb{Z} \text{ if } n = 0 \\ \mathbb{Z} \text{ if } n = 1 \\ 0 \text{ otherwise} \end{cases}$$

(b) If K consists of 5 edges consisting of a square and one diagonal then

$$H_n(K) = \begin{cases} \mathbb{Z} \text{ if } n = 0 \\ \mathbb{Z} \oplus \mathbb{Z} \text{ if } n = 1 \\ 0 \text{ otherwise} \end{cases} .$$

(c)

$$H_n(\partial\Delta^3) = \begin{cases} \mathbb{Z} \text{ if } n = 0 \\ \mathbb{Z} \text{ if } n = 2 \\ 0 \text{ otherwise} \end{cases}$$

There are a couple of extreme cases where the answer is obvious. The first is immediate from the definition.

Lemma 12.6. *If K is n-dimensional then*

$$H_n(K) := \ker(d : C_n(K) \longrightarrow C_{n-1}(K)).$$

12.3 Low dimensional homology

Recall that $\pi_0(K)$ is the set of equivalence classes of vertices under the equivalence relation generated by $v_0 \sim v_1$ when $\{v_0, v_1\}$ is an edge. We also showed that

$$\pi_0(K) \cong \pi_0(|K|)$$

using the map taking the equivalence class of a vertex to its path component.

Lemma 12.7.

$$H_0(K) = \mathbb{Z}[\pi_0(K)]$$

Proof. We note that $d_0 = 0$, and

$$d_1 : C_1(K) \longrightarrow C_0(K)$$

$$\langle v_0, v_1\rangle \longmapsto \langle v_1\rangle - \langle v_0\rangle$$

By definition therefore $H_0(K)$ is the quotient of $\mathbb{Z}[K_0]$ by the subgroup generated by differences of of vertices connected by edges. □

There is a similar statement for H_1, which we will not prove.

Theorem 12.8. *(Poincaré) If K is path connected and v_0 is a vertex of K then there is an isomorphism*

$$H_1(K) = \pi_1(|K|, v_0)^{ab}. \quad \square$$

12.4 Induced maps

We should really chack that simplicial homology gives a functor from simplicial complexes (and simplicial maps) to chain complexes (and chain maps).

Definition 12.9. *If $s : K \longrightarrow L$ is a simplicial map of abstract chain complexes then we may define $s_\sharp : C_k(K) \longrightarrow C_{jk}(L)$ by*

$$s\langle v_0, \ldots, v_k\rangle := \langle s(v_0), \ldots, s(v_k)\rangle.$$

Lemma 12.10. *(i) For any simplicial map $s : K \longrightarrow L$, the map $s_\sharp$ is well defined and taking all the components together we obtain a chain map*

$$s_\sharp : C_\bullet(K) \longrightarrow C_\bullet(L).$$

(ii) The construction is functorial in the senst that $(id_K)_\sharp = id_{C_\bullet(K)}$ and if $t : L \longrightarrow M$ is another map of chain complexes,

$$(t \circ s)_\sharp = t_\sharp \circ s_\sharp.$$

Proof. Autoproof. □

We may then take the induced map on homology

Definition 12.11. *If $s : K \longrightarrow L$ is a simplicial map of abstract chain complexes then we may define $s_* : H_k(K) \longrightarrow H_k(L)$ to be the map in homology induced by $s_\sharp$.*

Lemma 12.12. *The construction is functorial in the senst that $(id_K)_* = id_{H_*(K)}$ and if $t : L \longrightarrow M$ is another map of chain complexes,*

$$(t \circ s)_* = t_* \circ s_*.$$

Proof. Combine Lemmas 11.16 and 12.10 □

13 Chain homotopy, cones and spheres

In this section we will calculate the homology of spheres, and hence start to justify the slogan 'nth homology measures n-sphere-like holes'. In fact the work we do feeds into the more important project of showing that homology is an invariant of homotopy type.

Indeed, the main ingredient of both tasks is to see that the homology of certain obviously contractible complexes (cones) is the same as that of a point.

13.1 Chain homotopy

In fact chain complexes let one model even more of the geometric structure.

Definition 13.1. *If $\theta, \phi : A_\bullet \longrightarrow B_\bullet$ are two chain maps between chain complexes, a* chain homotopy *from θ to ϕ is a sequence of maps*

$$h_n : A_n \longrightarrow B_{n+1}$$

so that

$$dh + hd = \phi - \theta.$$

We then say that θ is homotopic to ϕ and write $\theta \simeq \phi$.

Lemma 13.2. *Chain homotopy is an equivalence relation on chain maps from $A_\bullet$ to $B_\bullet$.*

Proof. Exercise. □

Lemma 13.3. *If $\theta \simeq \phi$ then the induced maps on homology are the same: $\theta_* = \phi_*$.*

Proof. Suppose h is a chain homotopy, with $hd + dh = \phi - \theta$. Then for any n-cycle $z \in Z_n A$ we have

$$\phi_*[z] = [\phi z] = [\theta z + dhz + hdz] = [\theta z] = \theta_*[z].$$

□

Note that by the usual formal argument it follows that if two chain complexes are chain homotopy equivalent then they have isomorphic homology: in more detail if there are maps $\theta : C_\bullet \longrightarrow D_\bullet$ and $\phi : D_\bullet \longrightarrow C_\bullet$ so that $\theta \circ \phi \simeq id$ and $\phi \circ \theta \simeq id$ then θ_* and ϕ_* are inverse isomorphisms and $H_*(C) \cong H_*(D)$.

13.2 Cones

We may imagine the construction which takes a simplicial complex K in $\mathbb{R}^n$ and forms a new simplicial complex $c_P K$ in $\mathbb{R}^{n+1}$ where P is any point of $\mathbb{R}^{n+1}$ not in $\mathbb{R}^n$: the P-cone on K. This is obtained by joining all points of $|K|$ to P.

Definition 13.4. *(i) If $\sigma = \langle v_0, \ldots, v_k \rangle$ is a simplex in $\mathbb{R}^n$ and $P \in \mathbb{R}^{n+1} \setminus \mathbb{R}^n$ then we take $c_P\sigma = \langle P, v_0, \ldots, v_k \rangle$ this is a $(k+1)$-simplex in $\mathbb{R}^n$ (indeed, if $v_0, \ldots, v_k$ are $(k+1)$-points in general position in $\mathbb{R}^n$ then $P, v_0, \ldots, v_k$ are $(k+2)$-points in general position in $\mathbb{R}^{n+1}$).*

(ii) For the counterpart in the abstract setting, if P is not amongst the vertices of the set σ we take

$$c_P\sigma = \sigma \cup \{P\}.$$

(iii) If K is a simplicial complex in $\mathbb{R}^n$ and $P \in \mathbb{R}^{n+1} \setminus \mathbb{R}^n$ then $c_P K$ is the simplicial complex in $\mathbb{R}^{n+1}$ given by

$$c_P K = K \cup \{c_P\sigma \mid \sigma \in K\} \cup \{\langle P \rangle\}$$

(iv) If K is an abstract simplicial complex with vertex set V and P is not in V then $c_P K$ is the abstract simplicial complex with vertex set $V \cup \{P\}$

$$c_P K = K \cup \{c_P\sigma \mid \sigma \in K\} \cup \{\langle P \rangle\}$$

Lemma 13.5. *In both cases $c_P K$ is a simplicial complex, and $|c_P K|$ is the convex hull of P and $|K|$.*

Proof. It is clear that both constructions are closed under passage to faces, and (in the geometric setting) that two simplices intersect in a common face. □

First we observe that $|c_P K| \simeq *$. Indeed, the inclusion of P is a homotopy equivalence since the identity is homotopic to the constant map at P by the linear homotopy. The important thing is that we can mimic this algebraically: it is worth emphasizing that this fact is the key to showing that simplicial homology is a homotopy invariant.

Proposition 13.6. *For any simplicial complex K,*

$$H_*(c_P K) \cong H_*(pt)$$

(or, more explicitly, $H_0(c_P K) \cong \mathbb{Z}$ and $H_i(c_P K) = 0$ for $i \neq 0$).

Proof. We will show that the identity map of $C_\bullet(c_P K)$ is chain homotopic to the map ϵ_P defined as zero on $C_n(c_P K)$ if $n \neq 0$ and by $\epsilon_P\langle v\rangle = \langle P\rangle$ when $n = 0$. In other words, we need to find

$$h : C_n(c_P K) \longrightarrow C_{n+1}(c_P K)$$

for $n \geq 0$ so that

$$hd + dh = id - \epsilon_P.$$

Let us number P less than all the other vertices and if $\sigma = \langle v_0, \ldots, v_k\rangle$ we write $P\sigma = \langle P, v_0, \ldots, v_k\rangle$.

We may define h simplex by simplex, and if $P \notin \sigma$ we take

$$h(\sigma) = P\sigma, h(P\sigma) = 0.$$

Now we simply calculate. In degree 0 we have

$$(dh + hd)\langle v\rangle = d(\langle P, v\rangle) = \langle v\rangle - \langle P\rangle \text{ and } (dh + hd)\langle P\rangle = 0 = \langle P\rangle - \langle P\rangle.$$

When σ is a k-simplex for $k > 0$ we find

$$(dh + hd)(\sigma) = dP\sigma + hd\sigma = \sigma - Pd\sigma + Pd\sigma = \sigma$$

and

$$(dh + hd)(P\sigma) = d0 + h(\sigma - Pd\sigma) = P\sigma.$$

□

13.3 The homology of spheres

We have observed that $\partial\Delta^{n+1} = (\Delta^{n+1})^{(n)}$ is a simplicial version of the n-sphere S^n. We have already calculated its homology for $n = 0, 1, 2, 3$ from first principles and we are now equipped to calculate its homology in general.

Proposition 13.7. *For $n \geq 1$,*

$$H_k(\partial\Delta^{n+1}) = \begin{cases} \mathbb{Z} \text{ if } k = 0 \\ \mathbb{Z} \text{ if } k = n \\ 0 \text{ otherwise} \end{cases}$$

Proof. Note that Δ^{n+1} and $\partial\Delta^{n+1}$ are almost the same: the former just has one more simplex, than the latter. In particular

$$C_i(\Delta^{n+1}) = C_i(\partial\Delta^{n+1})$$

for $i \neq n+1$. This immediately shows

$$H_i(\partial\Delta^{n+1}) = H_i(\Delta^{n+1})$$

for $i \leq n-1$, which is as required by Proposition 13.6. For the interesting degree, n we may now argue

$$H_n(\partial\Delta^{n+1}) \overset{(1)}{=} Z_n(\partial\Delta^{n+1}) \overset{(2)}{=} Z_n(\Delta^{n+1}) \overset{(3)}{=} B_n(\Delta^{n+1}) \overset{(4)}{=} d(\Delta^{n+1}) \overset{(5)}{\cong} \mathbb{Z}.$$

For (1) we use Lemma 12.6. For (2) we use the fact that the two complexes agree below n. For (3) we use $H_n(\Delta^{n+1}) = 0$ by Proposition 13.6. For (4) we use the fact there is just one $(n+1)$-simplex. For (5) we use $H_{n+1}(\Delta^{n+1}) = 0$ by Proposition 13.6 so that the differential from C_{n+1} to C_n is injective. □