Part II

Simplicial complexes and Homology

The aim of this semester is to define another collection of invariants of spaces: the homology groups. These are easier to work with than the fundamental group because they are *abelian* groups. They are also easier to calculate. However they are a bit harder to define. The type of construction may also appear a little more unfamiliar.

The way the homology of a space X is defined is first to triangulate X (i.e., to find a combinatorial model for X (a *simplicial complex*) and then define homology groups of the combinatorial model. This is fairly easy, but it will be a long time before we show that we have defined *invariants* of X.

Anyway, we will spend quite a lot of time studying the homology groups of these combinatorial models. You should be reassured that they are interesing even in this context, so that this is not lost effort. With topological data analysis in vogue, finding invariants of combinatorial data is coming back into its own.

10 Simplicial complexes

10.1 Geometric simplicial complexes

Definition 10.1. *(i) The set of points* $\{v_0, v_1, \dots, v_k\}$ *in* $\mathbb{R}^n$ *are in* general position *if* $v_1 - v_0, v_2 - v_0, \dots, v_k - v_0$ *are linerly independent.*

(ii) If $\{v_0, v_1, \dots, v_k\}$ *in* $\mathbb{R}^n$ *are in general position then their convex hull*[1]

$$\langle v_0, v_1, v_2, \dots, v_k\rangle := \{\Sigma_{i=0}^k \lambda_i v_i \mid \Sigma_{i=0}^k \lambda_i = 1\}$$

is called the geometric simplex *with vertices* $v_0, v_1, \dots, v_k$.

(iii) The standard k*-simplex is*

$$\Delta^k = \langle e_0, e_1, \dots, e_k\rangle \subseteq \mathbb{R}^{k+1}$$

[1] A subset C of $\mathbb{R}^n$ is *convex* if the line segment $[P, Q]$ joining two points $P, Q \in C$ lies in C. The *convex hull* of a set X is the smallest convex set containing X.

where $e_0, e_1, \ldots, e_k$ *are the standard basis vectors (i.e.,* $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ *where the 1 is in the ith spot).*

Remarks 10.2. *(i) Evidently if* $\{v_0, \ldots, v_k\}$ *is in general position then* $k \leq n$. *It is not hard to check that whether or not a set is in general position does not depend on the order.*

(ii) The vertices of a geometric simplex are determined by the subset: they are the only points not lying on a line between two other points in the set.

Lemma 10.3. *If* $\{v_0, v_1, \ldots, v_k\}$ *in* $\mathbb{R}^n$ *are in general position and* $P \in \langle v_0, v_1, v_2, \ldots, v_k\rangle$ *then the numbers* λ_i *so that* $P = \Sigma_{i=0}^k \lambda_i v_i$ *and* $\Sigma_{i=0}^k \lambda_i = 1$ *are uniquely determined and* P *is said to have* barycentric coordinates $(\lambda_0, \lambda_1, \ldots, \lambda_k)$.

Definition 10.4. *The* barycentre *of* $\sigma := \langle v_0, \ldots, v_k\rangle$ *is the point with barycentric coordinates*

$$\hat{\sigma} = (\frac{1}{k+1}, \frac{1}{k+1}, \ldots, \frac{1}{k+1}).$$

Definition 10.5. *A* face *of a simplex* $\sigma = \langle v_0, \ldots, v_k\rangle$ *is the simplex spanned by a non-empty subset of its vertices.*

Definition 10.6. *Two simplices* $\sigma = \langle v_0, \ldots, v_k\rangle$ *and* $\tau = \langle w_0, \ldots, w_l\rangle$ *have good intersection if their intersection is empty or a face of each. Equivalently, they have good intersection if the intersection is simplex spanned by the intersection of their vertex sets:*

$$\sigma \cap \tau = \langle \{v_0, \ldots, v_k\} \cap \{w_0, \ldots, w_l\}\rangle.$$

Definition 10.7. *A* geometric simplicial complex *in* $\mathbb{R}^n$ *is a set* K *of simplices in* $\mathbb{R}^n$ *so that*

(i) Every face of a simplex in K *is a simplex in* K *and*

(ii) Any two simplices of K *have good intersection.*

Remarks 10.8. *A map of geometric simplices*

$$f : \langle v_0, \ldots, v_k\rangle \longrightarrow \langle w_0, \ldots, w_l\rangle$$

is linear function given by a function

$$f : \{v_0, \ldots, v_k\} \longrightarrow \{w_0, \ldots, w_l\}$$

on vertices. In other words

$$f\left(\sum_i \lambda_i v_i\right) = \sum_i \lambda_i f(v_i).$$

A map $f : K \longrightarrow L$ of geometric simplicial complexes is given by a map of geometric simplices for each simplex of K, in such a way that it is compatible with restriction to faces. In other words, it is specified by a map on vertex sets and takes simplices to simplices.

Definition 10.9. *(i) The* underlying space *of a geometric simplicial complex K is the union of its simplices:*

$$|K| = \bigcup_{\sigma \in K} \sigma.$$

(ii) A triangulation of a topological space X is a geometric simplicial complex K and a homeomorphism

$$t : |K| \xrightarrow{\cong} X.$$

Remarks 10.10. *Evidently $|K|$ is compact and Hausdorff, so these are necessary conditions for a space to have a triangulation.*

10.2 Abstract simplicial complexes

We've seen that simplices only depend on the vertices, which they determine. It is therefore natural to try and capture the structure of a simplicial complex purely in terms of them.

Definition 10.11. *(i) An abstract simplicial complex on a vertex set V is set K of subsets of V so that*

If $\emptyset \neq \tau \subseteq \sigma \in K$ then $\tau \in K$

The elements of K are called simplices. The dimension *of a simplex is one less than the number of vertices. If $\tau \subseteq \sigma$ we say that τ is a* face *of σ.*

(ii) A simplicial map $f : (K, V) \longrightarrow (L, W)$ between abstract simplicial complexes is a function $f : V \longrightarrow W$ on vertex sets so that if $\sigma \in K$ then $f(\sigma) \in L$.

This was supposed to be an abstraction of a geometric simplicial complex, so we define a functor

$$g : \text{AbstractSimplicialComplexes} \longrightarrow \text{GeometricSimplicialComplexes}$$

by taking $g(K, V)$ to be the geometric simplex in $\mathbb{R}V$ with simplices spanned by the basis vectors:

$$g(K, V) = \{\langle v_0, \ldots, v_k \rangle \mid \{v_0, \ldots, v_k\} \in K\}.$$

In the other direction, we define a functor

$$a : \text{GeometricSimplicialComplexes} \longrightarrow \text{AbstractSimplicialComplexes}$$

by taking $a(L)$ to be the abstract simplicial complex with the vertices of L as vertex set, and with the simplices being the vertices of those in L:

$$a(\sigma) = \text{Vertices of } \sigma.$$

Lemma 10.12. *(i) Given an abstract complex K, we have $ag(K) = K$*

(ii) Given an abstract complex L, we have $ga(L) \cong L$.

Remarks 10.13. *In the second statement, note that the original vertices of L have become the basis vectors of $\mathbb{R}[V]$, so $ga(L)$ will probably be in a different space.*

10.3 Examples

From one point of view simplicial complexes are just a way of putting coordinates onto something geometric. They are convenient for calculations, and with luck one can show that the answers are independent of the additional structure.

From a second point of view they are a convenient way of generating an enormous range of different topological spaces with interesting geoemtry and properties. The point is that you can specify a very complicated space with a finite amount of data.

Finally, they give ways of picking out geometry from complicated data. This final motivation has become very important recently with the advent of Big Data, and especially Topological Data Analysis (TDA). This is now a real industry....in the commercial sense that (for example) Ayasdi now employs over a hundred people.

Example 10.14. *(a) $|\Delta^n| \cong \overline{B}^n$.*

(b) $|\partial\Delta^n| \cong S^{n-1}$, where $\partial\Delta^n$ consists of all the proper faces of Δ^n.

(c) $|CrPo^{n-1}| \cong S^{n-1}$.

(d) Any compact surface admits a triangulation (Rado, 1925). Similarly for 3-manifolds (Moise, 1952)

(e) Any smooth manifold of dimension ≥ 4 has a triangulation (Cairns 1935 (triangulation), Whitehead 1940 (combinatorial)).

(f) In every dimension ≥ 4 there is a compact manifold with no tirangulation (Kirby-Siebenmann 1969 for combinatorial $n \geq 5$, later for arbitrary; Freedman, Casson for $n = 4$).

Example 10.15. *(i) One dimensional abstract simplicial complexes are the same as graphs (without loops or multiple edges).*

(ii)A graph determines a flag complex

$$\mathrm{Flag}(G) = \{\sigma \mid \text{ all edges of } \sigma \text{ are in } G\}.$$

For example

$$\Delta^n = \mathrm{Flag}(\text{Complete graph on } \{0, 1, \ldots, n\}).$$

Example 10.16. (Vietoris-Rips complex) *The idea here is to try and get some geometry out of a pattern of points in Euclidean space. This method is widely used to make sense of large data sets. They might actually be geometric (e.g., the set of positions of the observed galaxies on the celestial sphere) or something more abstract. This is one route in to Topological Data Analysis.*

Given a finite subset Q of $\mathbb{R}^n$, one may seek patterns.

For $\epsilon > 0$ we may define

$$VR_\epsilon(Q) = \{\sigma \subseteq Q \mid \mathrm{diam}(\sigma) \leq \epsilon\}$$

This is a simplicial complex, and we note $VR_\epsilon(Q)$ is always a flag complex.

It is also interesting to see what happens to $VR_\epsilon(Q)$ as ϵ grows. This leads to 'persistent' structures and to 'persistent homology' in particular.

Example 10.17. (Cech Nerve) *The idea here is that if we have a cover (or another set of subsets of a space) then this can tell us a lot about the space. Alternatively, different covers have rather different geometries, and some of the features are characteristic of the underlying space.*

If $\mathcal{U} = \{U_1, U_2, \ldots, U_n$ is a collection of subsets of a space then we may write

$$\check{C}(\mathcal{U}) = \{\sigma \subseteq A \mid \bigcap_{a \in \sigma} U_a \neq \emptyset\}.$$

It is clear that the Cech nerve is a simplicial complex.

We can describe examples like the cover of Euclidean space by regularly spaced balls. The largest number of balls that have non-trivial intersection is an interesting invariant of the cover, and this gives rise to one approach to covering dimension*, one of the ways of assigning a dimension to an arbitrary topological space.*

Example 10.18. *One example of this comes from a finite subset $Q \subseteq \mathbb{R}^n$, when we take $U_q = \overline{B}_\epsilon(q)$ and $\mathcal{U}_\epsilon = \{U_q \mid q \in Q\}$.*

The point of mentioning this is its relationship to the Vietoris-Rips complex. We see

$$\check{C}_\epsilon(Q) \subseteq VR_{2\epsilon}(Q)$$

since if $\sigma \in \check{C}_\epsilon(Q)$ then there is an $x \in \bigcap_{i\in\sigma} \overline{B}_\epsilon(q)$.

Both of these are heavily used in Topological Data Analysis.

Theorem 10.19. *If $\mathcal{U}$ is an open cover of X with the property that every intersection is empty or contractible then*

$$|\check{C}(\mathcal{U})| \simeq X.$$

Remarks 10.20. *Obviously the condition holds if all sets $U \in \mathcal{U}$ are convex subsets in $\mathbb{R}^n$ (since any intersection is convex).*

Corollary 10.21.

$$|\check{C}_\epsilon(Q)| \simeq \bigcup_{q\in Q} \overline{B}_\epsilon(q)$$

The proof is beyond the scope of the course, but here is the idea.

Form

$$|\check{C}(\mathcal{U})| \longleftarrow \coprod_\sigma U_\sigma \times |\sigma|/\sim \longrightarrow X$$

Both maps have contractible fibres (ie U_σ or $|\sigma|$). Now use the so-calle Vietoris-Begle theorem.

11 Homology

When defining homtopy invariants in topology we take geometric quantities (like loops) and identify them when they are geometrically related (by homotopy). In algebra the corresponding idea is to take an abelian group Z of possible values (like loops) and take the quotient by a subgroup B to obtain $H = Z/B$ (like loops mod homotopy).

11.1 Abelian groups

We pause to recall some basic abelian groups.

$$\mathbb{Z}, \mathbb{Z}/n \text{ for } n \geq 2.$$

In fact every group we need to consider will be a sum of these. The abelian groups A we consider are *finitely generated* in that there is a finite set of elements $a_1, a_2, \ldots, a_n$ so that every element a of A can be obtained from these by group operations:

$$a = \sum_i n_i a_i$$

for some integers n_i. Unlike bases of vector spaces, these integers are not uniquely determined.

Theorem 11.1. *Any finitely generated abelian group A is isomorphic to a direct sum of cyclic groups*

$$A = \mathbb{Z}/n_1 \oplus \mathbb{Z}/n_2 \oplus \cdots \oplus \mathbb{Z}/n_k,$$

and one can arrange

$$n_1 | n_2 | n_3 \cdots | n_k \text{ with } n_{k-r+1} = \cdots = n_k = 0$$

for some r. Thus

$$A = T \oplus F \text{ with } T \text{ a finite torsion group and } F = \mathbb{Z}^r.$$

Proof. The idea is that since A is finitely generated, there is a surjection $\theta : \mathbb{Z}^k \longrightarrow A$ for some A. Then $\ker(\theta)$ is a subgroup of a free abelian group and hence free (needs proof!). Thus we have

$$0 \longrightarrow \mathbb{Z}^l \xrightarrow{\theta} \mathbb{Z}^k \longrightarrow A \longrightarrow 0.$$

Thus θ is injective and $A = \mathbb{Z}^k / \operatorname{Im}(\theta)$. Now θ is represented by a $k \times l$ integer matrix, and by changing basis in both domain and codomain we can make the matrix diagonal. □

We remark that there are plenty of interesting abelian groups that are not finitely generated (such as $\mathbb{Q}$ or $\mathbb{R}$).

11.2 Exact sequences

Exact sequences have become a central element in the way topologists think. Indeed, this is reflected in the way the last proof was presented.

Definition 11.2. *If A, B and C are abelian groups, then we say that*

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C$$

is exact at B *if* $\ker(\beta) = \operatorname{Im}(\alpha)$.

Remark 11.3. In the above situation, we see that by the First Isomorphism Theorem,

$$\text{Im}\,(\beta) \cong B/\ker(\beta) = B/\,\text{Im}\,(\alpha).$$

Lemma 11.4. *(i) The sequence* $0 \longrightarrow A \stackrel{\alpha}{\longrightarrow} B$ *is exact at* A *if and only if* α *is injective.*

(ii) The sequenced $B \stackrel{\beta}{\longrightarrow} C \longrightarrow 0$ *is exact at* C *if and only if* β *is surjective.*

Example 11.5. *(a) The sequence* $\mathbb{Z} \stackrel{p}{\longrightarrow} \mathbb{Z} \stackrel{\pi}{\longrightarrow} \mathbb{Z}/p$ *is exact at the middle* $\mathbb{Z}$.

(b) The sequence $\mathbb{Z} \longrightarrow \mathbb{C} \longrightarrow \mathbb{C}^{\times}$ *is exact at* $\mathbb{C}$, *where the first map is inclusion and the second is* $z \mapsto e^{2\pi i z}$.

Definition 11.6. *(a) A sequence of abelian groups and homomorphisms*

$$\cdots \longrightarrow A_{n+1} \stackrel{\theta_{n+1}}{\longrightarrow} A_n \stackrel{\theta_n}{\longrightarrow} A_{n-1} \stackrel{\theta_{n-1}}{\longrightarrow} \cdots$$

is exact if each three term sequence

$$A_{n+1} \stackrel{\theta_{n+1}}{\longrightarrow} A_n \stackrel{\theta_n}{\longrightarrow} A_{n-1}$$

is exact at A_n.

(b) A short exact sequence *is an exact sequence of the form*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0.$$

We will see that short exact sequnces are the building blocks ot the analysis, so that understanding them is fundamental.

Lemma 11.7. *If*

$$0 \longrightarrow A \stackrel{\alpha}{\longrightarrow} B \stackrel{\beta}{\longrightarrow} C \longrightarrow 0.$$

is a short exact sequence, then

$$B/A' \cong C$$

where $A' = \text{Im}\,(\alpha) \cong A$. *In particular, if* A *and* C *are finite then*

$$|B| = |A| \cdot |C|.$$

Proof. By exactness at A, α is injective so that $A \cong \text{Im}\,(\alpha)$. By exactness at B, $\text{Im}\,(\alpha) = \ker(\beta)$. By exactness at C, β is surjective. The result now follows from the First Isomorphism Theorem. □

We give some examples to show that short exact sequences tell one a lot about the groups in them....but not quite everything.

Example 11.8. *(a) If we have an exact sequence*

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \longrightarrow C \longrightarrow 0,$$

then α is multiplication by n for some $n \neq 0$ and $C \cong \mathbb{Z}/n$.

(b) If

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow B \longrightarrow \mathbb{Z}/2 \longrightarrow 0,$$

is short exact then either $B \cong \mathbb{Z}/4$ or $B \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$.

(c) For any abelian groups A and C we can always form the trivial *short exact sequence*

$$0 \longrightarrow A \xrightarrow{\alpha} A \oplus C \xrightarrow{\beta} C \longrightarrow 0$$

where $\alpha(a) = (a, 0)$ and $\beta(a, c) = c$. The previous two examples show that not every short exact sequence is trivial.

Lemma 11.9. *Any exact sequence*

$$\cdots \longrightarrow A_{n+1} \xrightarrow{\theta_{n+1}} A_n \xrightarrow{\theta_n} A_{n-1} \xrightarrow{\theta_{n-1}} \cdots$$

may be broken up into short exact sequences

$$0 \longrightarrow \operatorname{Im}(\theta_{n+1}) \longrightarrow A_n \longrightarrow \ker(\theta_{n-1}) \longrightarrow 0.$$

11.3 Homology

The basic invariant of this sort of structure is homology.

Definition 11.10. *A chain complex is a sequence of abelian groups and group homomorphisms*

$$(A_\bullet, d_\bullet) = \left(\cdots \longrightarrow A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \xrightarrow{d_{n-1}} \cdots \right)$$

so that the composite of two adjacent maps is zero: $d_n \circ d_{n+1} = 0$.

Remark 11.11. (i) In a chain complex as above, we automatically have

$$\operatorname{Im}(d_{n+1}) \subseteq \ker(d_n).$$

(ii) We will often drop the subscript on d_n, leaving the context to determine it. Accordingly we write $(A_\bullet, d)$ (or even just $A_\bullet$) for the chain complex and the defining property is

$$d^2 = 0.$$

We note that an exact sequence is a chain complex, but not every chain complex is exact. The deviation from exactness is the homology.

Definition 11.12. *(i) Given a chain complex as above, we define the nth homology by*

$$H_n(A) = \frac{\ker(d_n : A_n \longrightarrow A_{n-1})}{\operatorname{Im}(d_{n+1} : A_{n+1} \longrightarrow A_n)}.$$

We sometimes write $H_*(A_\bullet) = \{H_n(A_\bullet)\}_n$ *for the collection of all homology groups.*

(ii) We sometimes refer to $Z_n A = \ker(A_n \longrightarrow A_{n-1})$ *as the group of* n-cycles *and* $B_n A = \operatorname{Im}(A_{n+1} \longrightarrow A_n)$ *as the group of* n-boundaries.

Remark 11.13. The homology $H_n(A_\bullet, d) = 0$ if and only if $(A_\bullet, d)$ is exact at A_n.

Example 11.14. *If* $A_n = 0$ *except for* $n = 0, 1$ *we automatically have a chain complex:*

$$0 \longrightarrow A_1 \xrightarrow{d_1} A_0 \longrightarrow 0.$$

We quickly see that $H_i(A) = 0$ *unless* $i = 0$ *or* 1*, and*

$$H_1(A) = \ker(d_1) \text{ and } H_0(A) = A_0/d_1(A_1).$$

One particular example of this may suggest the next section. We take $A_1 = \mathbb{Z}\{e_0, e_1, e_2\}$*,* $A_0 = \mathbb{Z}\{v_0, v_1, v_2\}$*, and*

$$d_1 e_0 = v_1 - v_0, d_1 e_1 = v_2 - v_1, d_1 e_2 = v_0 - v_2.$$

We quickly see

$$H_1(A) \cong \mathbb{Z} \text{ with generator } e_0 + e_1 + e_2 \text{ and } H_0(A) \cong \mathbb{Z} \text{ generated by } [v_0] = [v_1] = [v_2].$$

11.4 Chain maps

We have defined chain complexes, and these form a category with the appropriate structure preserving maps.

Definition 11.15. *If* $(A_\bullet, d^A)$ *and* $(B_\bullet, d^B)$ *are two chain complexes, a chain map*

$$\theta : (A_\bullet, d^A) \longrightarrow (B_\bullet, d^B)$$

is a sequence of maps

$$\theta_n : A_n \longrightarrow B_n$$

commuting with the differentials

$$d_n \circ \theta_n = \theta_{n-1} \circ d_n$$

for all n*. Pictorially, we have a commutative diagram*

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{d_{n+2}} & A_{n+1} & \xrightarrow{d_{n+1}} & A_n & \xrightarrow{d_n} & A_{n-1} & \xrightarrow{d_{n-1}} & \cdots \\
 & & \downarrow{\scriptstyle \theta_{n+1}} & & \downarrow{\scriptstyle \theta_n} & & \downarrow{\scriptstyle \theta_{n-1}} & & \\
\cdots & \xrightarrow{d_{n+2}} & B_{n+1} & \xrightarrow{d_{n+1}} & B_n & \xrightarrow{d_n} & B_{n-1} & \xrightarrow{d_{n-1}} & \cdots
\end{array}$$

We note that the identity is a chain map and the composite of two chain maps is a chain map, so that we have a category of chain complexes and chain maps.

Lemma 11.16. *If* $\theta : A_\bullet \longrightarrow B_\bullet$ *is a chain map then there is an induced map of homology groups*

$$\theta_* : H_n(A_\bullet) \longrightarrow H_n(B_\bullet)$$

defined by

$$\theta_*([z]) = [\theta_n(z)].$$

This is functorial in the sense that $id_* = id$ *and* $(\theta \circ \phi)_* = \theta_* \circ \phi_*$*.*

Proof. We start with $z \in \ker(d_n : A_n \longrightarrow A_{n-1})$. This then defines a homology class $[z]$. The first thing to check is that $\theta(z)$ is also a cycle. This follows from $d\theta z = \theta dz = \theta(0) = 0$.

Next, we check it does not depend on the coset representative. Indeed, if $[z] = [z']$ then $z' = z + dy$, and

$$\theta z' = \theta z + \theta dy = \theta z + d\theta y$$

and hence $[\theta z] = [\theta z']$ as required. □