ADELIC MODELS, RIGIDITY AND EQUIVARIANT COHOMOLOGY (CASE FOR SUPPORT)

J.P.C.GREENLEES

1. Track record

J. P. C. Greenlees is a leading world authority on equivariant cohomology theories and derived commutative algebra. He has been a leading figure and agenda-setter in the area of the project for over a decade as as is apparent from the listed publications.

Greenlees has held a Chair at the University of Sheffield since 1995; he was awarded the 1995 Berwick Prize of the London Mathematical Society. He has held visiting appointments at the University of Chicago and the Isaac Newton Institute (Cambridge). More recently, he was Research Professor at MSRI (Berkeley) for five months in 2014, and in 2015 he was Simons Researcher at CRM (Barcelona) (2 months) and Visiting Researcher at HIM (Bonn) (5 months).

His research is originated in equivariant algebraic topology and homotopy theory, but the connections with commutative algebra, derived algebraic geometry and representation theory that have emerged from it have become major themes in their own right.

He has coorganized a four month programme "New contexts for stable homotopy" at the INI (2002), a series of four meetings in Homotopy Theory in Oberwolfach, a meeting on the solution of the Kervaire Invariant problem at ICMS (Edinburgh) and is Principal Organizer of the 6 month programme "Homotopy Harnessing Higher Structures' scheduled for July 2018.

EPSRC supported projects include an extremely successful project on connections between local cohomology and algebraic topology (GR/M71350) finished in 2002 (for example [DG, DGI1, DGI2, DGI3, DGI4]). This work continues to be rather influential. The project Higher Structures on elliptic cohomology (EP/C52084X/1) finished in 2008, and was also very productive [AG, GH, ABGHR, GS, G8, G9, GS6, GS1, GS5, G10]. The project 'Orientability and Complete Intersections for Ring Spectra' (EP/E012957/1) finished in 2010, [BeG2, GHS, BGS, BeG3]. The project 'Rational equivariant cohomology theories' finished in 2014, the publications [GS6, G10, G11, BGKS, K1, K2, K3] layed the foundations for equivariant aspects of the present application.

Greenlees has supervised 10 postdoctoral students and 7 completed and 4 current PhD students.

2. Background

This proposal comes out of the study of equivariant cohomology theories for compact Lie groups G. Building on work in the 1990s [G1, G4, G3, G6], the PI proposed a vision explaining the entire structure in terms of the algebra and representation theory of the group G. There has been steady progress since [G7, G8, G9, GS1, GS5, GS6, G10, G11, BGKS] but recent work explaining how the subgroup structure can be recovered from structural properties [G13] has led to a conceptual upheaval, opening a new approach that encompasses a wide range of other examples. The purpose of this project is

- to develop these ideas
- to apply the ideas to extend the range of groups for which the the PI's conjecture is proved
- to apply the ideas to analyze similar structures from related areas
- to use the common structure to build bridges migrating structures from geometry into equivariant homotopy theory

We sketch this in more detail below.

2.A. **Some tensor triangulated categories.** In algebra, geometry and topology, we begin with some 'geometric' category $\mathcal{C}$ of objects of interest (spaces, varieties, modules, ...) and then consider homological invariants of them. This means that the invariants factor through an associated homotopy category $\overline{\mathcal{C}} = Ho(\mathcal{C})$, which therefore becomes of interest in its own right as some sort of universal invariant. Quite often $\mathcal{C}$ will have a tensor product compatible with other structure, and this will be inherited by $\overline{\mathcal{C}}$, so that $\overline{\mathcal{C}}$ is a *tensor triangulated category.*

There are innumerable examples, but as something to hold on to

- *commutative algebra:* We start with a commutative Noetherian ring R and consider the category $\mathcal{C}$ of chain complexes of R-modules. This example guides us and suggests the language. and $\overline{\mathcal{C}}$ is the derived category of R-modules
- *representation theory:* We start with a finite group G, and consider the category $\mathcal{C}$ of kG-modules for a field k, and then $\overline{\mathcal{C}}$ is the stable module category
- *algebraic geometry:* We begin with a variety or scheme X and consider the category $\mathcal{C}$ of complexes of quasi-coherent modules over the structure sheaf and its derived category $\overline{\mathcal{C}}$.
- *algebraic topology:* We begin with a commutative ring spectrum R and take $\mathcal{C}$ to be the category of R-modules, and $\overline{\mathcal{C}}$ is its homotopy category (also often called the derived category by analogy with the commutative algebra example). Chromatic examples and equivariant examples are the central subjects in this project, and will be described in more detail below.

The general methods apply to all cases, but the guiding principle of the proposal is that the language and ideas of commutative algebra should be used to give a complete algebraic model of equivariant cohomology theories.

2.B. **Rigidity.** Working with $\overline{\mathcal{C}}$ passes through several stages of evolution. To start with one needs tools for working in $\overline{\mathcal{C}}$ (constructions, methods of calculation). With experience one wants to move to a more global view considering general properties of $\overline{\mathcal{C}}$, such as its small (=compact) objects, or its thick subcategories of small objects. Finally, to achieve complete enlightenment one may want to understand the entire category $\overline{\mathcal{C}}$.

At this final stage one may want to build a model of $\overline{\mathcal{C}}$ that one understands completely, probably by finding an algebraic category $\mathcal{C}' = \mathcal{C}_{alg}$ so that $\overline{\mathcal{C}}' = Ho(\mathcal{C}')$ is actually equivalent to $\overline{\mathcal{C}} = Ho(\mathcal{C})$. To do this one needs a rigidity theorem, stating that any two categories $\mathcal{C}, \mathcal{C}'$ with a tensor triangulated homotopy category looking like $\overline{\mathcal{C}}$ are in fact equivalent. The less we need to know about the homotopy categories to deduce the equivalence of $\mathcal{C}$ and $\mathcal{C}'$, the stronger the result.

To show this is more than philosophy, we sketch a couple of examples. We wil then return to a more systematic account of the project.

2.C. **Local rigidity.** We begin with the building block, which corresponds to the one-point case of the global phenomenon (a whole variety of points) that we are intereted in.

The example comes from equivariant topology. For definiteness we consider the circle group $\mathbb{T}$ and we consider cohomology theories on free $\mathbb{T}$-spaces. In fact the category $\overline{\mathcal{C}}$ of such $\mathbb{T}$-equivariant cohomology theories on $\mathbb{T}$-free spaces is the homotopy category of the category $\mathcal{C}$ of free $\mathbb{T}$-spectra. To give a more accessible problem we consider those theories whose values are rational vector spaces.

The value of such a cohomology theory on a contractible free $\mathbb{T}$-space is a torsion module over the polynomial ring $H^*(B\mathbb{T}) = \mathbb{Q}[c]$, so there is a functor

$$\overline{\mathcal{C}} \longrightarrow \text{tors-}\mathbb{Q}[c]\text{-modules},$$

and in fact one may prove that $\overline{\mathcal{C}}$ is actually *equivalent* to the derived category of torsion $\mathbb{Q}[c]$-modules. One way to prove this is to recognize that $\mathbb{Q}[c]$ is *intrinsically formal* in the sense that any differential graded algebra A with $H_*(A) = \mathbb{Q}[c]$ is quasi-isomorphic to $\mathbb{Q}[c]$. It then follows that $\mathcal{C}$ itself is Quillen equivalent to the category of differential graded torsion $\mathbb{Q}[c]$-modules [GS1, GS5] and the statement about derived categories follows.

Finally, from this it is clear there is a unique thick subcategory of small objects, corresponding to the fact we should think of this as a situation with 'only one closed point'.

2.D. **The Balmer spectrum.** We next explain the statement about closed points.

Associated to a tensor triangulated category $\overline{\mathcal{C}}$ we may consider prime ideals $\wp$ in the subcategory of small objects (i.e., thick subcategories, closed under tensoring with an arbitrary object and so that if $x \otimes y \in \wp$ then $x \in \wp$ or $y \in \wp$). The *Balmer spectrum* $\mathrm{Spc}(\overline{\mathcal{C}})$ consists of all prime ideals in the category $\overline{\mathcal{C}}_{small}$ of small objects; in the first instance it has the structure of a partially ordered set (poset) under inclusion. The minimal (sic) elements are the closed points.

For example if $\overline{\mathcal{C}}$ is the derived category of R-modules for a commutative Noetherian ring R then Hopkins showed that $\mathrm{Spc}(R)$ is in natural (order-reversing) bijection with the usual prime spectrum $Spec(R)$. Thus the Balmer minimal primes correspond to maximal ideals in the classical sense, and to closed points in the usual sense of algebraic geometry.

Example 2.1. The Balmer spectrum of free rational $\mathbb{T}$-spectra above has indeed got just one point.

Example 2.2. On the other hand we may take $\mathcal{C}$ to consist of all rational G-spectra for a torus G, so that $\overline{\mathcal{C}} = Ho(G\text{-spectra})$ consists of *all* rational G-equivariant cohomology theories.

One may then prove [G13] that $\mathrm{Spc}(\overline{\mathcal{C}})$ corresponds to the partially ordered set $\mathrm{Sub}_a(G)$ of closed subgroups (where $K \leq H$ if K is a subgroup of H and K/H is a torus). Indeed the prime corresponding to the closed subgroup K consists of all spectra whose geometric K-fixed points are contractible:

$$\wp_K = \{X \mid \Phi^K X \simeq_1 *\}.$$

Half of the result follows from the Borel-Hsiang-Quillen Localization Theorem together with idempotents in the Burnside ring, but the algebraic model of [G8] is used to finish the harder part.

The point of this example is that the poset $\mathrm{Sub}_a(G)$ has been recovered from the formal tensor triangulated structure. This striking fact led the PI to a complete adjustment of his conceptual framework, and hence to the present proposal.

2.E. **Global rigidity.** Returning to the discussion of rigidity, we now consider the category $\mathcal{C}$ of rational $\mathbb{T}$-spectra. We have learnt that $\mathrm{Spc}(\overline{\mathcal{C}}) = \mathrm{Sub}_a(\mathbb{T})$, and one may build an algebraic model $\mathcal{C}_{alg}$ from $\mathrm{Sub}_a(\mathbb{T})$. This is the category of differential objects in the abelian category $\mathcal{A}(G)$ sketched below.

The PI has conjectured that for an arbitrary compact Lie group G, the homotopy category of rational G-spectra is equivalent to the derived category of $\mathcal{A}(G)$:

$$\overline{\mathcal{C}} = \mathrm{Ho}(G\text{-spectra}/\mathbb{Q}) \simeq D(\mathcal{A}(G)) = \overline{\mathcal{C}}_{alg}.$$

and indeed that the equivalence comes from a Quillen equivalence

$$\mathcal{C} = G\text{-spectra}/\mathbb{Q} \simeq dg\mathcal{A}(G) = \mathcal{C}_{alg}.$$

The conjecture has been proved for the rank 1 groups $G = SO(2), O(2), SO(3)$ and for tori in [G4, G3, G6, Ba1, Ba2, K1, K2, K3, G8, GS6], with partial results in general [GS1, GS5, G11].

To give the flavour, and to tie in with the project, we need to very loosely sketch the construction of $\mathcal{A}(G)$. The full description is elaborate and would distract us (see [G8, G9, G10] for further information). The data over a subgroup $K \in \mathrm{Sub}_a(G)$ corresponds to the classification of G/K-equivariant cohomology theories for free G/K-spaces, which were classified in [GS5] along the lines of Subsection 2.C. Taking this more slowly, we construct a sheaf $\mathcal{O}$ of rings on $\mathrm{Sub}_a(G) = \mathrm{Spc}(\overline{\mathcal{C}})$. The stalk at K is the polynomial ring $H^*(BG/K)$. There is additional structure relating these stalks corresponding to the Localization Theorem. Now the algebraic model $\mathcal{A}(G)$ consists of certain sheaves of $\mathcal{O}$-modules.

Finally, we should describe the proof of rigidity. The idea is that G-spectra are modules over the equivariant sphere spectrum $\mathbb{S}$, and it is proved in [GS6] that $\mathbb{S}$ is the pullback of a punctured $(r+1)$-cube of other simpler ring spectra, themselves built from spectra corresponding to subgroups. For each individual subgroup one may use the one-point rigidity argument, and one checks that these arguments fit together to give a global rigidity statement.

This process is precisely analogous to the process by which the ring of integers $\mathbb{Z}$ is a pullback of the Hasse square, and hence built from $\mathbb{Q}$ and the p-adic integers $\mathbb{Z}_p^\wedge$ for maximal ideals (p).

2.F. **The proposal.** The proof sketched in the previous subsection[GS6] works specifically in the category of G-spectra, and uses special features of that case. The core of the present proposal is to develop the general machinery needed to attempt something similar for a more general tensor triangulated category. It is well known that not all tensor triangulated categories are rigid, so it is fundamental to provide workable criteria for rigidity.

There are four classes of example of immediate interest.

- Rational equivariant cohomology theories for compact Lie groups G other than a torus
- chromatic homotopy theory
- (derived) algebraic geometry.

It also seems likely that there may be useful insights even the case of classical commutative algebra, with a higher dimensional Hasse theory.

Notwithstanding the development of general machinery and the illumination it will bring, the central application of this proposal, and the one by which the effectiveness of the tools should be judged, is that of rational equivariant cohomology theories for a general compact Lie group G.

3. General formalism

We begin with a monoidal model category $\mathcal{C}$, and associated tensor triangulated homotopy category $\overline{\mathcal{C}}$, and Balmer spectrum $\mathfrak{X} = \mathrm{Spc}(\overline{\mathcal{C}})$. We will describe the general construction, in the knowledge that this applies to modules over Noetherian rings and to rational G-spectra over a torus. One of the purposes of the project is to identify the appropriate level of generality and give useful sufficient conditions under which the bald assertions here make sense, and then to provide useful criteria for when they are actually true.

3.A. **Localization and completion.** If $\wp$ is a prime, under mild hypotheses there is an associated localization $L_\wp : \overline{\mathcal{C}} \longrightarrow \overline{\mathcal{C}}$ and a completion $\Lambda_\wp : \overline{\mathcal{C}} \longrightarrow \overline{\mathcal{C}}$ [GM1, DG]. For suitable model structures, these are defined as Quillen functors at the level of model categories, and the corresponding functors on model categories also have universal properties.

Viewing $L_\wp$ and $\Lambda_\wp$ as functors of $\wp$, we first note that

$$L_\bullet : \mathrm{Spc}(\overline{\mathcal{C}}) \longrightarrow [\overline{\mathcal{C}}, \overline{\mathcal{C}}]$$

is covariant (in the Balmer order!), whereas

$$\Lambda_\bullet : \mathrm{Spc}(\overline{\mathcal{C}}) \longrightarrow [\overline{\mathcal{C}}, \overline{\mathcal{C}}]$$

is contravariant.

If $\mathcal{C}$ is a category of R-modules and $L_\wp$ is smashing (as is the case in numerous examples of interest), $L_\wp$ takes values in $L_\wp R$-modules. This in turn gives a filtration of $\mathrm{Spc}(R)$ by the subspectra $\mathrm{Spc}(L_\wp R)$ of primes lying inside $\wp$, and it is not surprising that one can assemble a model from this local data. We may call this the *local model*.

On the other hand, we would like to assemble a model from objects supported at a single prime. There are two equivalent models of this (as in [DG]), one corresponding to torsion modules and one to complete modules, and in any case this part is viewed as simple. For example this corresponds to free and cofree G/K-spectra or to p-torsion or p-complete abelian groups.

The way these are stuck together is a generalization of the Tate square [GM2] in equivariant homotopy theory or for the Hasse square in ordinary commutative algebra (see also [BBT] for an approach in algebraic geometry). The primitive case is discussed in an appropriate style in [G2].

Problem 3.1. (a) Give a good formal context for discussing $L_\wp$ and $\Lambda_\wp$.
(b) Fit the equivalence between torsion modules and complete modules into this framework.

There are a number of treatments of these matters in general terms [GM1, DG, G12, BHV]. The aim is to select a convenient set of details and give a systematic development of the theory. The test of the details is in the following two problems, the first of which should be straightforward.

Problem 3.2. Describe a local model $L_\bullet \mathcal{C}$, and give a Quillen equivalence $\mathcal{C} \simeq L_\bullet \mathcal{C}$.

Problem 3.3. (a) Describe a general system of Hasse-Tate squares for assembling the model at each prime into a *global adelic* model $\mathcal{C}_{gad}$.
(b) Under appropriate hypotheses to be identified, relate $\mathcal{C}$ to $\mathcal{C}_{gad}$ by a chain of Quillen functors.

3.B. **Adelic cohomology.** When it comes to criteria for rigidity, it will be necessary to assume rigidity at each point (for example if the relevant stalks of $\mathcal{O}$ are polynomial). However there are generally obstructions to assembling the stalks properly. The crudest obstruction is a cohomological invariant we describe here. The commutative algebra case and the case of rational torus-equivariant cohomology shows this invariant has extremely interesting conceptual and calculational content.

Using the data, we can define a cochain complex to this situation with

$$C^s(\mathrm{Spc}(\overline{\mathcal{C}}); M) = \pi_*(\prod_{\wp_0} L_{\wp_0} \prod_{\wp_1 \subset \wp_0} \cdots \prod_{\wp_{s-1} \subset \wp_{s-2}} L_{\wp_{s-1}} \prod_{\wp_s \subset \wp_{s-1}} L_{\wp_s} \Lambda_{\wp_s} M).$$

For example with $R = \mathbb{Z}$ the Hasse square

$$\begin{array}{ccc} (\mathbb{Z}) & \longrightarrow & \mathbb{Q} \\ \downarrow & & \downarrow \\ \prod_p \mathbb{Z}_p^\wedge & \longrightarrow & \mathbb{Q} \otimes \prod_p \mathbb{Z}_p^\wedge \end{array}$$

corresponds to the cochain complex

$$(C^0 \longrightarrow C^1) = \left(\prod_p \mathbb{Z}_p^\wedge \times \mathbb{Q} \longrightarrow \mathbb{Q} \otimes \prod_p \mathbb{Z}_p^\wedge \right)$$

Since the Hasse square is both a pullback and a pushout, the adelic cohomology is $\mathbb{Z}$ in degree 0 and 0 in degree 1. More generally one may show that if $\overline{\mathcal{C}}$ is the derived category of a Noetherian ring R then the adelic cohomology is entirely in degree 0, where it is R.

The adelic cohomology is not always just in degree 0. To start with, if it is applied to complex curves, the use of residues shows that adelic cohomology calculates sheaf cohomology of the curve. In the case arising from rational torus-equivariant homotopy theory this gives an algebraic distillation of the (rationalized) Segal-tom Dieck splitting:

$$H^s(\mathrm{Spc}(G\text{-spectra}); \mathbb{S}) = \bigoplus_{\mathrm{cod}(K)=s} H_*(BG/K).$$

The proof combines the Cousin complex with an interesting combinatorial analysis of the subgroup structure of G in an illuminating way.

Problem 3.4. Give a systematic account of adelic cohomology, and provide a wide range of calculations.

It is natural to expect adelic cohomology to play a significant role in giving criteria for equivalences.

Problem 3.5. Develop a general theory of adelic models based on $\mathfrak{X}$, together with an obstruction theory for constructing comparison functors.

Problem 3.6. Assuming that one has local rigidity at each prime of $\mathrm{Spc}(\overline{\mathcal{C}})$ develop criteria for when these globalize.

We now turn to a range of examples to which we wish to apply the theory. The value of the general machinery will be judged on its effectiveness in the examples.

3.C. **Equivariant cohomology theories.** For a compact Lie group G, one may consider the category $\mathcal{C}$ of rational G-spectra, with associated homotopy category $\overline{\mathcal{C}}$ of rational G-equivariant cohomology theories. This is the principal class of examples for the current project, partly because the calculational challenges are more easily manageable.

Problem 3.7. Establish the precise relationship between $\mathrm{Spc}(\overline{\mathcal{C}})$ and the poset of closed subgroups under cotoral inclusion with the f-topology of [G1].

This is understood for tori and rank 1 groups, and one should expect to start with examples (groups with identity component a torus, $SU(3)$, ...). The solution is likely to be achieved in two stages as for the torus in [G13]. Showing the conjugacy classes in the topological category is a subobject of $\mathrm{Spc}(\overline{\mathcal{C}})$ is quite accessible. This is used as the basis for a model, allowing one to bootstrap a full answer.

Restricting attention to primes corresponding to subgroups (which is all of $\mathrm{Spc}(\overline{\mathcal{C}})$ if the conjecture is true) we can already define the adelic cohomology, and this is certainly an interesting invariant. The expectation is that the realization of the geometric version of the cochain complex is in fact the sphere, so that the adelic cohomology will give a spectral sequence for the calculation of its homotopy.

Problem 3.8. Calculate the adelic cohomology of the sphere for a general group. Relate it to the Segal-tom Dieck splitting, and decide if the adelic spectral sequence collapses.

Even in the case of a torus mentioned above this has substantial algebraic content. Again, one would expect to start with examples (rank 1 is an easy exercise from the model and rank ≤ 2 is not difficult).

One of the appeals of the adelic model is it explains how to assemble a model from information supported at individual primes. In the case of the circle group, the role of this was played by the torsion model [G4]. This has higher homological dimension than the standard model but the building blocks are easier to understand.

Problem 3.9. Using the global adelic model, identify the torsion model for a general torus.

The problem as stated should not be hard, but the investigation of the homological properties of the torsion model will be more interesting. One might conjecture that the torsion model has injective dimension $2r$, but this has only been checked in detail for $r = 1$.

The other major appeal of this approach is that the model is in the same form as one coming from algebraic geometry. In [G7] equivariant elliptic cohomology is constructed by an ad hoc importation of information from algebraic geometry. As a result of this project, the category of sheaves over an elliptic curve will itself have a global adelic model, and a minor transposition will give an object in the global adelic model of equivariant spectra.

Problem 3.10. Develop a systematic route for migration between contexts, and apply it to migrate a range of algebraic geometric objects into equivariant topology.

The only fully known example is an elliptic curve C [G7]. There is a clear expectation for the case of C^r for an elliptic curve C, but it should be thoroughly codified. Going beyond this, the problem is effectively infinite in extent. For the present project, we would only expect a few basic examples.

3.D. **Chromatic homotopy theory.** Quillen established the bridgehead by which the theory of formal groups could be applied to stable homotopy theory, but thanks to the classification of thick subcategories of small spectra by Devinatz-Hopkins-Smith (essentially the calculation of the Balmer spectrum of the stable homotopy category) we know this is intrinsic structure.

Chromatic homotopy theory is highly developed, and we will not attempt a full introduction here. For the purposes of this project the point is that with $\mathcal{C}$ being the category of spectra, we have a much less algebraic example of our context, but one where the massive investment of effort in the past should still enable us to answer some questions. Apart from classical approaches, the work of T.Barthel and collaborators ([BHV], and other work in preparation) will be relevant.

Problem 3.11. (a) Relate the global adelic model to existing approaches to chromatic homotopy theory, including the place of the chromatic convergence theorem and the chromatic splitting conjecture.

(b) Identify the adelic cohomology in this setting.

Rigidity theorems in chromatic homotopy theory rely on small primes and great complexity to give rigidity, so general globalization results seem unlikely, even if the present machinery brings useful insights.

3.E. **Schemes.** Rather than investigating commutative algebra in non-Noetherian contexts, the priority will be to see how Noetherian commutative algebra engenders geometry. The vanishing of higher adelic cohomology for affine Noetherian schemes suggests that adelic cohomology simply gives an approach to ordinary sheaf cohomology, giving access to standard machinery. Our main philosophy is to see the algebraic context as well understood and to take advantage of the wealth of examples.

Problem 3.12. Establish conditions under which adelic cohomology calculates sheaf cohomology.

In order to use the migration route we need to be able to identify a range of interesting objects.

Problem 3.13. Identify a useful class of rigid schemes for migration.

3.F. **Representation theory.** Suppose that G is a finite p-group and k is a field of characteristic p. The category of kG-modules is equivalent to the category of torsion modules over the commutative ring spectrum $C^*(BG;k)$ [DG, DGI1]. The Balmer spectrum of $C^*(BG)$-modules agrees with that of $H^*(BG)$-modules, although $C^*(BG)$ and $H^*(BG)$ are usually not quasi-isomorphic. This suggests that process of assembly will be of interest in this case.

The category of stable kG-modules (finitely generated modules modulo projective modules) agrees with modules over the Tate ring spectrum t_G in the sense of [GM2]. This corresponds to finitely generated $C^*(BG)$-modules modulo torsion modules. By the BGG correspondence we should think of this as 'coherent sheaves over $\mathrm{Proj}(C^*(BG))$'. But in any case at the level of Balmer spectra this is $\mathrm{Proj}(H^*(BG))$. Once again the difference between the invariants of $C^*(BG)$ and $H^*(BG)$ is of interest.

4. Summary

The proposal is to set up a systematic formal framework for building models for homotopy categories as sheaves of modules over sheaves of rings on the Balmer spectrum as a topological category. An associated adelic cohomology theory will be defined and studied. This will be applied in a range of examples to include modules over a commutative Noetherian ring, certain sheaves over schemes and examples from chromatic homotopy theory.

However, the most important example consists of rational equivariant cohomology theories, which will be studied in their own right. In particular they will be shown to be rigid in a range of cases extending those for which this is already known, and examples will be imported from the other cases.

5. National Importance

Topology, algebra and algebraic geometry (the main topics of the proposal) are core parts of mathematics. They underpin large parts of mathematical infrastructure, but also play a significant role through applied topology and data science (topological data analysis), in mathematical physics and elsewhere. They are also a particular area of UK strength, as detailed in the EPSRC Landscape documents, and one with strong links to other European centres (Barcelona, Bonn, Copenhagen, ...). To ensure the UK remains in a position to link to and benefit from this nearby strength following Brexit we need to maintain capacity.

Sheffield is one of the UK centres of algebraic topology with a unique position representing equivariant homotopy, chromatic homotopy, homotopical commutative algebra and connections to modular representation theory. It is particularly important that the established strength is renewed by the flow of newly trained PhDs. These strengths flow into the UK mathematical infrastructure through strong links to Oxford, Cambridge, Aberdeen, Southampton, Durham and Glasgow.

6. Academic beneficiaries and Visiting Fellows.

6.A. **Algebraic topology.** The construction of accessible algebraic models of any important class of objects is a significant advance. A small algebraic model brings methods of calculation, analysis and construction. The methods developed to make this progress (for example in the theory of model categories) are likely to be of a type with wider significance.

Perhaps more valuable is the conduit built to import ideas and structures from algebraic geometry. This would bring in rich and highly developed methods, bringing to mind the great chromatic insights from the theory of formal groups.

6.B. **Algebraic geometry.** The classical Hasse Principle is a valuable local to global principle. Local to global principles in many forms are extremely useful and routinely used. The one described here, building the whole derived category from information purely at individual points gives a systematic account of well known ways of working and puts it in a much more general context.

6.C. **Representation theory.** It is hoped that the perspectives gained from this point of view within modular representation theory of finite groups will be of interest, but the comparison with the more calculationally accessible case of rational equivariant cohomology theories seems even more likely to be useful.

6.D. **Expertise of Visiting Fellows.** The fellows will bring expertise from three fields fundamental to the project.

Paul Balmer (UCLA). An expert in a wide range of contexts including algebraic geometry and representation theory, and especially in the general study of tensor triangulated categories.

Clark Barwick (MIT) An expert in abstract homotopy theory and ∞-categories. Great fluency in the use of quasi-categorical techniques in topology, K-theory and algebraic geometry.

David Benson (Aberdeen). An expert in modular representation theory, the classification of thick and localizing categories and algebraic examples generally.

B. Shipley (UIC) is an expert on Quillen model categories. She is a collaborator of the PI's on a closely related project, and it is hoped that she can continue to participate directly in this project. Furthermore, she is ideally suited to explain the model theoretic ingredients to the RA.

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